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Track Coverage in Sensor Networks

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Abstract— **So far coverage problems have been formulated to address area coverage or to maintain line-of-sight visibility in the presence of obstacles (i.e., art-gallery problems). Although sensor networks often are employed to track moving targets, none of the existing formulations deal with the problem of allocating sensors in order to achieve track-formation capabilities over a region of interest. This paper investigates the problem of finding the configuration of a network with n sensors such that the number of tracks intercepted by k sensors is optimized without providing redundant area coverage over the entire region. This problem arises in applications where proximity sensors are employed that have individual detection capabilities, and that obtain limited measurements from each track, possibly at different moments in time. By assuming that the target travels along a straight unknown path, and that the sensors are omnidirectional with limited range (i.e., their visibility can be represented by a circle), it can be shown that the tracks detected by one or more (k) sensors always are contained by a coverage cone. Therefore, the track coverage of the network can be measured through the opening angle of the coverage cone and formulated in terms of unit vectors that depend on the sensors' range and location. Through this approach, the coverage of a given network configuration can be rapidly assessed. Also, a coverage function is obtained that, when maximized with respect to the sensor location, optimizes the number of tracks detected over a rectangular area of interest. The same approach can potentially be applied to other convex polygons and to three-dimensional Euclidian space.**

I. INTRODUCTION

In sensor networks literature, coverage typically refers to the problem of *area coverage*, that is, ensuring that every point in a two-dimensional space is within the range of at least one sensor in the sensor network (e.g., [9]). Depending on the underlying physics, the area coverage of one sensor is the area of a circle or sector centered at the sensor location. Then, the network coverage can be investigated by considering the union of all the areas covered by its sensors. Another well-known formulation of coverage is the *art-gallery problem*, where a point or sensor sees the target if the line segment between them does not intersect any obstacles (also known as line-of-sight visibility) [10]-[12]. This problem is concerned with placing the sensors such that the targets in a given area of interest that includes obstacles are in the line-of-sight of at least one of the sensors. Although very useful in many sensor applications, none of the existing formulations address coverage as it pertains to target tracking by means of multiple sensors. This paper presents a novel coverage problem, referred to as *track coverage*, that can be formulated using elementary planar geometry, and can be used to deploy sensors such that the probability of track detection is maximized.

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II. BACKGROUND ON TARGET TRACKING BY MEANS OF MULTIPLE SENSORS

The problem of target tracking by means of distributed sensors arises in many applications, such as, surveillance systems, tracking of endangered species, and manufacturing, and consequently it has received considerable attention. *Tracking* refers to the estimation of the state (e.g., position, velocity, and acceleration) of a moving object through one or more sensors positioned on stationary or moving platforms. After a new target is detected by one or more sensors in search mode its track is formed. That is, from that moment onward, its state trajectory is estimated from a set of measurements that are associated to it. In a sensor network the data also is fused to maintain the track as precisely as possible. The problems of data association and data fusion that arise in the tracking process have been studied through several approaches that include the Nearest Neighbor (NN) algorithm, Probabilistic Data Association (PDA), Multiple Hypothesis Tracking (MHT), and assignment [1]-[6]. Typically, after the data is associated with each target and fused by one of these algorithms, it is used together with past observations to estimate the target state by means of well-known Kalman-filter equations [7], as reviewed in [3]. However, these methods rely on frequent measurements obtained from sensors that can observe the target over the same time interval, such as air-traffic-control radars.

In order for sensor networks to be practical and affordable, simple proximity sensors often are employed in those applications where there is no *a-priori* knowledge of the target track. In proximity networks the measurements are very limited and may be collected by the sensors at different times, while the target moves across the sensor network. In this case, the event-based algorithm developed in [8] can be used to determine the potential track of a single target based on multiple reports on its location. The target is assumed to move at constant speed and heading through the sensor field maintaining a constant source amplitude. Also, each sensor reports its location in two-dimensional space and a single value of received signal at the sensor-to-target closest-point-of-approach (CPA), denoted by $e^{\text{CPA}} = e(t^{\text{CPA}})$. But, the CPA time, t^{CPA}, is not reported to the central position and, thus, cannot be accounted for in the analysis. Since in the proximity networks of interest tracking would be performed through this event-based algorithm, these assumptions are extended to the coverage problem presented in this paper.

In addition, the track-forming algorithm in [8] assumes that the received signal is isotropic energy attenuated by the environment according to the following power law,

$$e_i(t) = cFr_i^{-\alpha}(t) \tag{1}$$

which constitutes the model for the i^{th} proximity sensor. Sensors that measure magnetic, acoustic, or optical waves, for example, can be represented by (1) assuming linear wave propagation models. Thus, the value of the attenuation coefficient α and of the scaling constant c depend on the physical mechanism of wave propagation and on the environment. The target source level F is independent of time and of the sensor location. Therefore, by letting $S = (cF)^{-1/\alpha}$, the measurements of the range-to-CPA can be represented by the variable,

$$\beta_i = (e_i^{\text{CPA}})^{-1/\alpha} = Sr_i(t^{\text{CPA}}) > 0 \tag{2}$$

for the i^{th}-sensor in the proximity network. Under the simplifying assumptions above, these measurements contain the same scaling factor or invariant S, which is independent of the target detected and may be unknown.

It is shown in [8] that given a set of $\beta_i = Sr_i(t^{\text{CPA}})$ measurements for sensors located at $\mathbf{x}_i$, the track of a target with constant speed and heading is a straight line that is jointly tangent to all circles defined as $C_i(S) \equiv \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_i\| \leq \beta_i/S\}$, for a fixed value of S. Where, $\mathbf{x} \in \mathbf{R}^{2\times 1}$ and $\|\cdot\|$ denotes the *Eucledian* or *quadratic* norm. For example, if two proximity sensors are located respectively at $\mathbf{x}_1$ and $\mathbf{x}_2$ and have each reported a detection, the corresponding circle C_i can be drawn with radius $r_i = \beta_i/S$, where β_i are the measurements from the i^{th} sensor and $i = 1, 2$, as shown in Fig. 1. Then, the potential tracks, also illustrated in Fig. 1, can be obtained without knowledge of S [8]. There are four potential tracks that correspond to two sensor detections: an interior and an exterior track, and their corresponding reflections about the axes that connects $\mathbf{x}_1$ and $\mathbf{x}_2$ (the reflections are not shown in Fig. 1 for simplicity). Even from this simple example it can be seen that two or more sensor detections may be required for reliable tracking. On the other hand, multiple detections can be obtained without covering the entire area surrounding the target-track of interest, since the detections can take place at different times along the target trajectory.

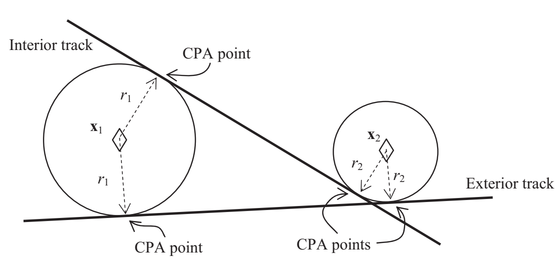

Fig. 1. Geometry of potential interior and exterior tracks formed by the CPA events of two sensors (symbolized by $\diamondsuit$), located at $_1$ and $_2$ (adapted from [8]).

III. TRACK COVERAGE PROBLEM

The track coverage problem consists of finding the locations of n sensors that maximize the number of tracks that are intercepted by at least k sensors (where, $k \leq n$), and that cross a pre-defined area of interest. For simplicity, the problem is formulated based on the following assumptions: (*i*) the targets move at constant speed and heading across the area (i.e., tracks can be represented by straight lines); (*ii*) the area of interest is a rectangle; (*iii*) the visibility or range of each sensor can be represented by a disk centered at the sensor location. Extensions to maneuvering tracks, and to other shapes of the area-of-interest and sensor range (such as, polygons and sectors, respectively) are all subjects of future work.

The set of all tracks in two-dimensional space can be described by the equation representing a straight line in the x_1x_2-plane, i.e., $x_2 = ax_1 + b$, with slope a and intercept b. According to Section II, a CPA detection event takes place when the track path is tangential to a circle of radius r_i and centered at the i^{th} sensor location, $\mathbf{x}_i = [x_{1,i} \quad x_{2,i}]^T$. Letting $\mathbf{x}_0$ denote the position of the intercept between the track path and the x_2-axes, as illustrated in Fig. 2, the position of the i^{th} sensor can be expressed more conveniently with respect to the track intercept:

$$\mathbf{v}_i \equiv (\mathbf{x}_i - \mathbf{x}_0) = \begin{bmatrix} x_{1,i} \\ (x_{2,i} - b) \end{bmatrix} \tag{3}$$

Hence, given that the CPA point must be within range of the sensor, it can be easily shown that the tracks detected are those whose slope and intercept satisfy the following equation,

$$r_i = \left| \frac{(b + ax_{1,i} - x_{2,1})}{\sqrt{a^2 + 1}} \right| \leq r_i^{max} \tag{4}$$

where, r_i^{max}, is the maximum range of the i^{th} sensor, and is assumed to be known for $\forall i$ (a brief proof is provided in Appendix I).

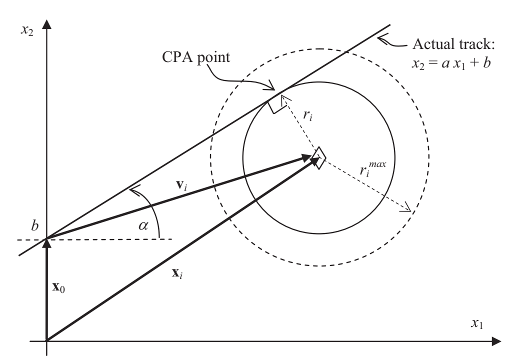

Fig. 2. Geometry of one-sensor one-track detection, where the equation $x_2 = ax_1 + b$ describes the track, and the sensor with maximum range r_i^{max} is located at $_i$ in the x_1x_2-plane.

A. *Assessing track coverage for a known configuration*

For a given sensor network configuration, in which all the sensor positions and maximum ranges are known, (4) can be used to determine the number of CPA detection events for a set of tracks crossing the region of interest. As an example, consider a representative subset of the tracks with intercept $b \in [x_2^{min}, \quad x_2^{max}]$ and slope $a = tan(\alpha)$, with $\alpha \in (-\pi/2, \quad \pi/2)$. This subset can be obtained by discretizing these parameters through intervals δb and $\delta\alpha$, respectively, thus choosing values in the set:

$$\{x_2^{min} : \delta b : x_2^{max}\} \times \{(-\pi/2 + \delta\alpha) : \delta\alpha : (\pi/2 - \delta\alpha)\} \quad (5)$$

Equation (4) is evaluated for every pair (a, b) in (5) and, using Boolean logic, the number of sensors capable of intersecting the corresponding target can be determined. Let A be a matrix with all the values of a that correspond to the set (5), and B a matrix with all the values of b in (5). Then, (4) can be used to compute a matrix R_i containing the CPA radius r_i of the tracks defined by the pair (A, B), for the i^{th} sensor. A logical array or *truth table*, with 1 representing true and 0 representing false, can be obtained from the following operation

$$D_i = \{R_i \leq r_i^{max}\} \quad (6)$$

indicating whether the i^{th} sensor has made a detection (true) or not (false). In a sensor network where at least k sensors must detect the moving target in order to positively form its track path, the logical array,

$$P = \left\{ \sum_i D_i \leq k \right\} \quad (7)$$

indicates whether each track defined by the pair (A, B) has had sufficient coverage in the present sensor network.

Although this approach can be used to assess the track coverage of a given sensor network configuration, a proper coverage function must be derived in order to maximize it with respect to the sensors locations. This problem is the subject of the next section.

IV. CONSTRUCTION OF TRACK-COVERAGE FUNCTION

The track coverage of a known configuration of sensors can be assessed by evaluating (4) for a subset of track paths traversing the region of interest, as explained in the previous section. However, in order to solve the track coverage problem that maximizes the number of tracks intercepted by the sensor network (Section III), a function expressing the coverage with respect to the sensor location must be determined. Clearly, the number of straight lines that traverse an area is infinite. Moreover, in order to be determined, the track path must be intercepted by at least k of the n sensors. The approach presented in this paper constructs the desired coverage function by introducing the concept of coverage cone for one sensor and with respect to one reference axes. This step allows to quantify the amount of tracks intercepted using a finite metric. Secondly, intersection cones are used to represent the amount of tracks intercepted by k sensors, with respect to their location and one reference axes. Finally, the coverage function for an area of interest in the shape of a rectangle is constructed by considering the intersection cones in reference to four axes that are each aligned with one side of the rectangle. This approach can then be extended to any polygon by induction.

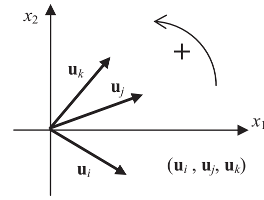

Fig. 3. Example of three vectors ordered according to the orientation of the x_1x_2-plane, i.e., ($_i$, $_j$, $_k$).

A. *Track coverage of one sensor*

Consider a sensor that is indexed by i and is located at $\mathbf{x}_i$ in the x_1x_2-plane, as illustrated in Fig. 4. For a given x_2-intercept denoted by b, and located at $\mathbf{x}_0 = [0 \quad b]^T$, the set of tracks that are potentially intercepted by the sensor with a maximum range r_i^{max} (which hereon will be denoted simply by r_i) is contained by a so-called two-dimensional coverage cone which can be defined by two unit vectors $\hat{\mathbf{l}}_i$ and $\hat{\mathbf{h}}_i$ (Fig. 4). In this paper, vectors are said to be *ordered* according to the orientation of the reference frame. An example is provided in Fig. 3. In this problem, the order $(\hat{\mathbf{l}}_i, \hat{\mathbf{h}}_i)$ indicates that if these vectors are translated such that their origins coincide and $\hat{\mathbf{l}}_i$ is rotated through the smallest angle possible to meet $\hat{\mathbf{h}}_i$, this rotation is in the same direction as the orientation of the x_1x_2-plane (in this case, positive in the counterclockwise direction). With this order, $\hat{\mathbf{l}}_i$ and $\hat{\mathbf{h}}_i$ define the directions of the *lowest* and *highest* tracks that can be intercepted by the sensor, respectively. Given the intercept b, every track that can be intercepted by the sensor will lie inside the coverage cone, and every track that cannot be intercepted will lie outside the coverage cone.

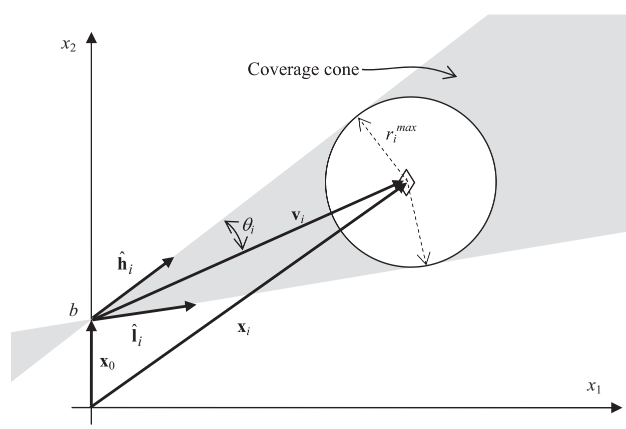

Fig. 4. Coverage cone and corresponding unit vectors, $\hat{\mathbf{l}}_i$ and $\hat{\mathbf{h}}_i$, for a sensor located at $_i$ and an intercept b.

Let $\mathbf{v} \equiv (\mathbf{x}_i - \mathbf{x}_0)$ denote the position of the sensor relative to the x_2-intercept b, and let θ_i denote half the opening angle of the cone (Fig. 4). Then, the unit vectors defining the cone can be obtained through rotation matrices, as follows,

$$\hat{\mathbf{h}}_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \equiv \mathbf{Q}_i^+ \hat{\mathbf{v}}_i \tag{8}$$

and,

$$\hat{\mathbf{l}}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix} \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \equiv \mathbf{Q}_i^- \hat{\mathbf{v}}_i \tag{9}$$

where, $\mathbf{Q}_i^- = (\mathbf{Q}_i^+)^T$. Also, since the lowest and highest tracks both form right triangles with $\mathbf{v}_i$ and with the radius of the circle C_i, the following trigonometric relationships,

$$\sin\theta_i = \frac{r_i}{\|\mathbf{v}_i\|} = \frac{r_i}{\sqrt{x_{1,i}^2 + (x_{2,i} - b)^2}} \tag{10}$$

and,

$$\cos\theta_i = \frac{\sqrt{\|\mathbf{v}_i\|^2 - (r_i)^2}}{\|\mathbf{v}_i\|} \tag{11}$$

can be used to express the rotation matrices with respect to the sensor location, $\mathbf{x}_i$.

It can be easily seen from this example (Fig. 4) that the amount of tracks within range of the i^{th} sensor can be maximized by maximizing the angle θ_i or, equivalently, $\sin\theta_i$. For a single sensor and a single value of the x_2-intercept b, the trivial solution to this maximization problem is to place the sensor at $\mathbf{x}_0$, such that all possible tracks through $x_2 = b$ can be intercepted. Using these simple constructs, however, a function can be obtained to express the track coverage defined in Section III with respect to the positions of n sensors, that can then be determined by numerical optimization. The next step is to address the tracks covered by the i^{th} sensor with respect to a range of intercepts on the x_2 axes, i.e., $b \in [x_2^{min}, \quad x_2^{max}]$. For simplicity, this range is discretized by letting $b_{\ell+1} = b_\ell + \delta b$, such that the track coverage function for the axes can be obtained by summing over b_ℓ:

$$\mathcal{T}_{x_2}(\mathbf{x}_i) \equiv \sum_\ell \sin\theta_i(b_\ell) \tag{12}$$

$$= \sum_{b_\ell = x_2^{min}}^{x_2^{max}} \frac{r_i}{\sqrt{x_{1,i}^2 + (x_{2,i} - b_\ell)^2}} \tag{13}$$

In principle, the function above could be utilized to express the coverage in the entire x_1x_2-plane, since all of the straight lines that lie in this plane must intersect the x_2 axes. However, this would comprise letting the limits in the summation go to $\pm\infty$, which clearly is not a practical solution. Moreover, track coverage typically is desired over an area of interest in the shape of a polygon, often a rectangle. Then, in order for (12) to include the majority of the tracks traversing a rectangle, a very large b-range would have to be considered, which would lead to also including a large number of tracks that never intersect this area. Thus, an approach for addressing track coverage over a rectangular region is developed in Section IV-C. In the next section, a track coverage function is derived for multiple (k) sensors.

B. Track coverage of multiple sensors

It was shown in [8] that CPA events from at least two sensors are necessary to determine potential track path locations from perfect β-measurements, through the methodology reviewed in Section II. Depending on the application, when error measurements and other uncertainties are factored into the problem of track determination, it may be desirable to have more than two detections. Hence, the coverage of $k \geq 2$ sensors in a network of size n must be addressed such that a track path can be considered to be detected (and subsequently determined) when it is intercepted by at least k sensors during the time it takes the target to travel through the area of interest.

Given the x_2-intercept b_ℓ, the set of tracks intercepted by k sensors is contained by a so-called *k-coverage cone* that can be determined from the coverage cones of the individual sensors. Let I be the index set of the list of position vectors, $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, for k sensors in the network, and let the ordered pair of unit vectors $(\hat{\mathbf{l}}_i, \hat{\mathbf{h}}_i)$ define the coverage cone of the sensor located at $\mathbf{x}_i$, with $i \in I$. Also, let all unit vectors $\hat{\mathbf{l}}_i$, $i \in I$, be ordered according to the orientation of the x_1x_2-plane, with $\hat{\mathbf{l}}^*$ denoting the last member of the ordered list. This is equivalent to the following statement,

$$\sin\gamma_l = \sup\{\sin\gamma_i, i \in I\} \equiv \sin\gamma^* \tag{14}$$

where, $\hat{\mathbf{l}}_i = [\cos\gamma_i \quad \sin\gamma_i]^T$. Similarly, let all unit vectors $\hat{\mathbf{h}}_i$, $i \in I$, be ordered according to the orientation of the x_1x_2-plane, and denote the first member of the ordered list by $\hat{\mathbf{h}}^*$, i.e.,

$$\sin\lambda_j = \inf\{\sin\lambda_i, i \in I\} \equiv \sin\lambda^* \tag{15}$$

where, $\hat{\mathbf{h}}_i = [\cos\lambda_i \quad \sin\lambda_i]^T$. Then, the ordered pair $(\hat{\mathbf{l}}^*, \hat{\mathbf{h}}^*)$ defines the k-coverage cone for the sensors positioned at $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. This cone is illustrated for two sensors in Fig. 5. The opening angle ψ (at the vertex) of the cone is the magnitude of the cross product between the two unit vectors

$$\sin\psi = \|\hat{\mathbf{l}}^* \times \hat{\mathbf{h}}^*\| \tag{16}$$

Using (8)-(9), this angle can be written in terms of the sensors locations,

$$\sin\psi = \begin{vmatrix} (\hat{\mathbf{l}}^*)^T \\ (\hat{\mathbf{h}}^*)^T \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{v}}_l^T \mathbf{Q}_l^+ \\ \hat{\mathbf{v}}_j^T \mathbf{Q}_j^- \end{vmatrix} \tag{17}$$

or, more explicitly, as illustrated in Appendix II. Where, $|\cdot|$ is the matrix determinant.

Thus, the number of tracks intercepted by all k sensors can be maximized by positioning the sensors such that ψ or, equivalently, $\sin\psi$ are maximized. The k-coverage function can be defined as,

$$\mathcal{T}_{x_2}(\mathbf{x}_l, \mathbf{x}_j) = \sum_{b_\ell = x_2^{min}}^{x_2^{max}} H(\psi)\sin\psi, \quad l, j \in I \tag{18}$$

subject to (14) and (15), and with $\sin\psi$ given by (17). The Heaviside function $H(\psi) = H(\sin\psi)$ guarantees that if the

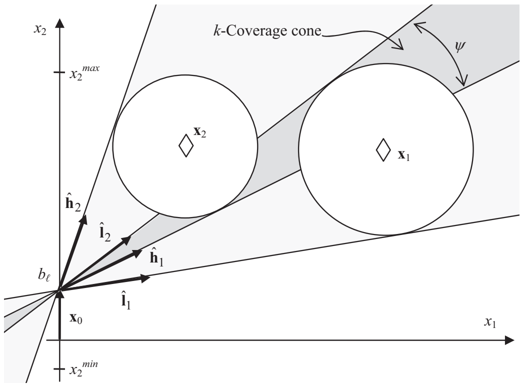

Fig. 5. 2-Coverage cone of two sensors located at $_1$ and $_2$, with intercept $b_\ell \in [x_2^{min}, \quad x_2^{max}]$.

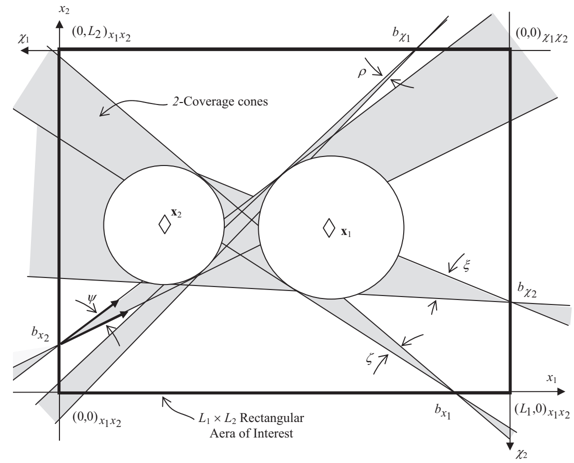

Fig. 6. 2-Coverage-cone definition illustrated for two sensors located at $_1$ and $_2$, and a rectangular area of interest, $\mathcal{A}$, with vertices $(0,0)_{x_1x_2}$, $(0,0)_{\chi_1,\chi_2}$, $(L_1,0)_{x_1x_2}$, and $(0,L_2)_{x_1x_2}$.

order of $\hat{\mathbf{l}}^*$ and $\hat{\mathbf{h}}^*$ is reversed, the coverage is zero. k-track coverage for a rectangular area can be derived by considering the k-track coverage of the network with respect to each axes that can be aligned with a side of the rectangle, as explained in the following section.

C. k-Track Coverage over a Rectangular Area

Consider the track coverage problem over a rectangular area $\mathcal{A}$ in a two-dimensional plane. As illustrated in Fig. 6, place the x_1x_2-frame of reference along two sides of the rectangle, such that its origin coincides with the lower-left vertex, labeled as $(0,0)_{x_1x_2}$. Also, place a second frame of reference along the remaining sides of the rectangle, namely the $\chi_1\chi_2$-frame, such that its origin coincides with the vertex opposite to $(0,0)_{x_1x_2}$, labeled as $(0,0)_{\chi_1,\chi_2}$. The remaining vertices can be labeled as $(L_1,0)_{x_1x_2}$ and $(0,L_2)_{x_1x_2}$, as shown in Fig. 6, such that L_1 and L_2 are the width and height of the rectangle, respectively. Every track traversing this region intercepts two sides of the rectangle. Also, it intercepts two and only two of the axes in the x_1x_2- and $\chi_1\chi_2$- frames, and these intercepts must fall within the following intervals: $b_{x_1}, b_{\chi_1} \in [0 \quad L_1]$ and $b_{x_2}, b_{\chi_2} \in [0 \quad L_2]$, where the subscript denotes the axes intercepted. Therefore, using the approach described in the previous section, the k-coverage cones can be formulated with respect to each of these four axis, and summed over the corresponding intercepts to include all possible tracks.

Equation (12) expresses the k-track coverage with respect to the x_2-axes, with $x_2^{min} = 0$ and $x_2^{max} = L_2$. The track function with respect to the x_1-axes is determined by defining the vector $\mathbf{x}_0$ to be the location of the intercept b_{x_1}, i.e., $\mathbf{x}_0 = [b_{x_1} \quad 0]^T$. Then, let $\mathbf{v}_i = (\mathbf{x}_i - \mathbf{x}_0)$, where the sensor position $\mathbf{x}_i$ must be expressed in the same coordinate frame, namely x_1x_2. The unit vectors defining the k-coverage cone are obtained through the same procedure outlined in Section IV-B. The opening angle ζ can be obtained from the magnitude of their cross product, and the coverage function for the x_1-axes can be written as

$$\mathcal{T}_{x_1}(\mathbf{x}_l, \mathbf{x}_j) = \sum_{b_{x_1}=0}^{L_1} H(\zeta)\sin\zeta, \quad l, j \in I \tag{19}$$

Since the plane has the same orientation as in Section IV-B, the indices l, j are determined from the same equations, (14) and (15), for every value of the intercept b_{x_1}. An equation for $\sin\zeta$ written explicitly with respect to the sensors locations is provided in Appendix II.

The same approach is then applied to axis χ_1 and χ_2 in Fig. 6. By choosing a plane with the same orientation as x_1x_2 (i.e., positive in the counterclockwise direction), the unit vectors defining the k-coverage cones with respect to these axis are ordered in the same manner and, thus, also can be obtained through the relationships (14) and (15). For each axes considered (χ_1, then χ_2), the vector $\mathbf{x}_0$ is defined as the position of the intercept along that axes, and the vector $\mathbf{v}_i = (\mathbf{x}_i - \mathbf{x}_0)$ is recomputed based on this intercept. Subsequently, the unit vectors $\hat{\mathbf{h}}_i$ and $\hat{\mathbf{l}}_i$, which define the lowest and highest track for each sensor $i \in I$, are obtained from (8)-(11), and are ordered according to (14)-(15) to obtain the two unit vectors describing the k-coverage cone, $\hat{\mathbf{h}}^*$ and $\hat{\mathbf{l}}^*$. Then, the k-track coverage function for the entire region of interest can be obtained by summing the contribution of each axes, i.e.,

$$\mathcal{T}_{\mathcal{A}}(\mathbf{x}_l, \mathbf{x}_j) = \textstyle\sum_{b_{x_2}=0}^{L_2} H(\psi)\sin\psi + \sum_{b_{x_1}=0}^{L_1} H(\zeta)\sin\zeta + \sum_{b_{\chi_2}=0}^{L_1} H(\xi)\sin\xi + \sum_{b_{\chi_1}=0}^{L_2} H(\rho)\sin\rho \tag{20}$$

subject to (14)-(15). Although the coverage problem is formulated by means of two reference frames, the sensors coordinates, $x_{1,i}$ and $x_{2,i}$, can be expressed with respect to a single reference frame using a simple transformation. For example, in this case $(\mathbf{x}_i)_{\chi_1\chi_2} = [(-x_{1,i} - L_1) \quad -x_{2,i}]^T$, where $x_{1,i}$ and $x_{2,i}$ refer to the x_1x_2-plane. The equations for each opening angle in the coverage function can be found in Appendix II.

V. Results and Applications

The methodology developed in this paper can be used to assess the coverage of a sensor network configuration with respect to an axes or an area of interest. In particular, Section

III-A introduces a simple method for considering a subset of all the possible tracks of interest, and computing which of these tracks are detected or missed by a sensor network of known position and range. The k-coverage of a network of $n = 8$ sensors, with ranges and configuration illustrated in Fig. 7, is considered for illustration. In this case, $k = 3$ CPA detections are required for a positive track detection. The tracks that are missed by this network are computed using (4), and are plotted in Fig. 7. By the same approach, using (6) it also is possible to compute the number of detections with respect to the parameter space, which consists of the slope and intercept of the tracks, as shown in Fig. 8.

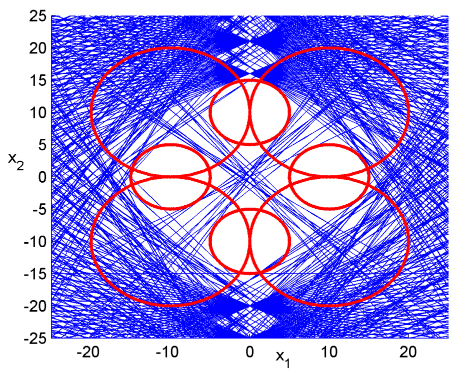

Fig. 7. Sample of tracks missed by an initial sensor network configuration with the ranges r_i illustrated by the corresponding circles (C_i), and $k = 3$ CPA detections required for track coverage.

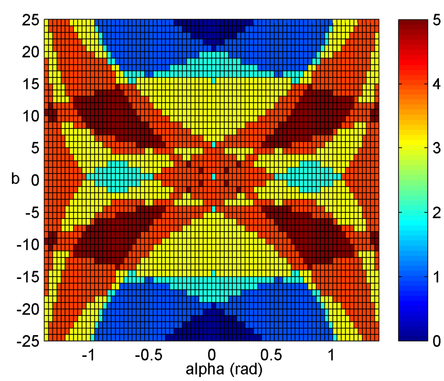

Fig. 8. Number of detections (colorbar) with respect to track parameters for the sensor network configuration in Fig. 7.

Now, suppose the configuration is altered, for example as would happen for sensors floating and drifting in the ocean, and suppose the new configuration can be determined (e.g., by GPS). Also, the sensor performance, and thus the range, may vary across the network due to changing environmental conditions. Then, the k-coverage of the network can be re-computed using the new sensor positions and ranges, as illustrated through an example in Fig. 9. This figure shows the new sensor characteristics and the tracks missed by the new configuration, based on the requirement that $k = 3$ CPA detections are needed for positive track detection. The value of k is application specific, and can be considered as a design parameter. Figure 10 shows that the number of detections with respect to the parameter space has decreased overall, and that new holes are beginning to form in the k-coverage of the network. The analytical functions derived in Section IV can also be used for measuring and assessing coverage, as well as for computing the probability of detection over an area of interest. But, most importantly, by expressing the k-coverage explicitly with respect to the sensors locations and ranges, they can be used to compute the optimal network configuration for a given area of interest.

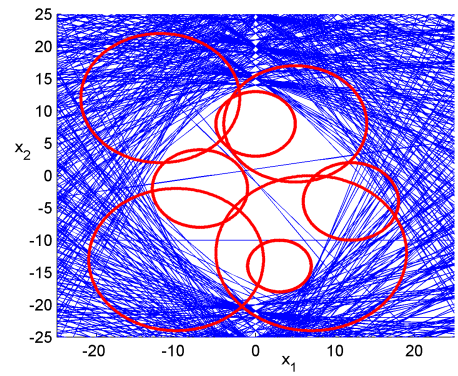

Fig. 9. Sample of tracks missed by a non-uniform sensor network configuration with the ranges r_i illustrated by the corresponding circles (C_i), and $k = 3$ CPA detections required for track coverage.

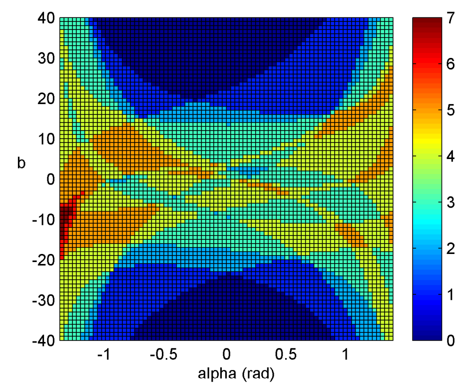

Fig. 10. Number of detections (colorbar) with respect to track parameters for the sensor network configuration in Fig. 9.

VI. CONCLUSIONS

A novel approach is presented for defining and formulating coverage in systems where multiple sensors track a moving target through limited measurements, such as closest-point-of-approach (CPA) detections. The approach is based on planar geometry and on the introduction of a so-called k-coverage cone, which allows to express the amount of tracks detected by k sensors in a network in terms of opening angles along the boundaries of a rectangular region of interest. A technique for rapidly assessing the coverage of a given network configuration, with known sensor ranges and locations, is derived and demonstrated through simulations involving eight sensors and $k = 3$ required detections for track formation. Also through this approach, a coverage function is derived that expresses the k-track coverage analytically in terms of the sensor locations and ranges. This result allows to formulate the k-track coverage as an optimization problem, where the coverage function is to be maximized with respect to the sensor network configuration.

APPENDIX I
PROOF OF EQUATION (4)

Consider the CPA triangle formed by joining the x_2-intercept, b, the CPA point, and the sensor position in the x_1x_2-plane (as shown in Fig. 2). This is always a right triangle, where the side opposite to the right angle is the vector $\mathbf{v}_i = (\mathbf{x}_i - \mathbf{x}_0)$, with $\mathbf{x}_0 = [0 \quad b]^T$. Let $\mathbf{w} = [-b/a \quad -b]^T$ be a vector parallel to the straight line, $x_2 = ax_1 + b$, representing a track path detected by the i^{th} sensor. Then, the angle θ_i that is opposite to the right angle at the CPA point, can be obtained from the following dot product,

$$\mathbf{v}_i \cdot \mathbf{w} = \|\mathbf{v}_i\|\|\mathbf{w}\| \cos\theta_i = \frac{-bx_{1,i}}{a} - b(x_{2,1} - b) \quad (21)$$

and the CPA radius is given by

$$r_i = \|\mathbf{v}_i\| \sin\theta_i \quad (22)$$

Taking the ration between (22) and (21) leads to,

$$\frac{r_i}{-b(\frac{x_{1,i}}{a} + x_{2,i} - b)} = \frac{\tan\theta_i}{\|\mathbf{w}\|} = \frac{\tan\theta_i}{|\frac{b}{a}|\sqrt{a^2+1}} \quad (23)$$

Using the trigonometric relationship,

$$\tan(\theta_i + \alpha) = \frac{\tan\theta_i + \tan\alpha}{1 - \tan\theta_i \tan\alpha} = \frac{(x_{2,i} - b)}{x_{1,i}} \quad (24)$$

and, observing that $a = \tan\alpha$, an equation for $\tan\theta_i$ is found solely with respect to the track parameters a and b:

$$\tan\theta_i = \frac{(x_{2,i} - b - ax_{1,i})}{(x_{1,i} + ax_{2,i} - ab)} \quad (25)$$

Hence, by combining (25) with (23) and simplifying the result, an equation can be obtained expressing the CPA radius in terms of the track parameters, i.e.,

$$r_i = \left| \frac{(b + ax_{1,i} - x_{2,1})}{\sqrt{a^2+1}} \right| \leq r_i^{max}$$

where, the CPA radius must be within the sensor range, r_i^{max}, in order for the track to be detected.

APPENDIX II
OPENING ANGLES EQUATIONS

Using the equations in Section IV-B, the k-coverage opening angle with respect to the x_2-axes can be formulated as:

$$\begin{cases} \sin\psi = \frac{1}{m_l^2 m_j^2}\{[x_{1,l}q_l + (x_{2,l} - b_\ell)r_l][x_{1,j}r_j + (x_{2,j} - b_\ell) \\ \cdot q_j] - [x_{1,j}q_j - (x_{2,j} - b_\ell)r_j][(x_{2,l} - b_\ell)q_l - x_{1,l}r_l],\} \\ m_i \equiv \|\mathbf{v}_i\| = \sqrt{x_{1,i}^2 + (x_{2,i} - b_\ell)^2}, \\ q_i \equiv \sqrt{m_i^2 - r_i^2}, \forall i \in I \end{cases} \quad (26)$$

Similarly, it can be shown (Section IV-C) that the k-coverage opening angle with respect to the x_1-axes is

$$\begin{cases} \sin\zeta = \frac{1}{m_l^2 m_j^2}\{[(x_{1,j} - b_{x_1})q_j + x_{2,j}r_j][(x_{1,l} - b_{x_1})r_l + \\ x_{2,l}q_l] - [(x_{1,l} - b_{x_1})q_l - x_{2,l}r_l][x_{2,j}q_j - (x_{1,j} - b_{x_1}) \\ \cdot r_j]\};\ m_i \equiv \|\mathbf{v}_i\| = \sqrt{(x_{1,i} - b_{x_1})^2 + x_{2,i}^2}, \\ q_i \equiv \sqrt{m_i^2 - r_i^2}, \forall i \in I \end{cases} \quad (27)$$

By applying the usual procedure to the remaining axis, χ_1 and χ_2, it can be easily shown that,

$$\begin{cases} \sin\xi = \frac{1}{m_l^2 m_j^2}\{[(L_1 - x_{1,l})q_l - (x_{2,l} + b_{\chi_2})r_l] \\ \cdot[(L_1 - x_{1,j})r_j - (x_{2,j} + b_{\chi_2})q_j] - [(L_1 - x_{1,j})q_j \\ +(x_{2,j} + b_{\chi_2})r_j][(x_{1,l} - L_1)r_l - (x_{2,l} + b_{\chi_2})q_l]\}, \\ m_i \equiv \|\mathbf{v}_i\| = \sqrt{(L_1 - x_{1,i})^2 + (x_{2,i} + b_{\chi_2})^2}, \\ q_i \equiv \sqrt{m_i^2 - r_i^2}, \forall i \in I \end{cases} \quad (28)$$

and,

$$\begin{cases} \sin\rho = \frac{1}{m_l^2 m_j^2}\{[(L_1 - b_{\chi_1} - x_{1,l})q_l - x_{2,l}r_l] \\ \cdot[(L_1 - b_{\chi_1} - x_{1,j})r_j - x_{2,j}q_j] - [(L_1 - b_{\chi_1} - x_{1,j})q_j \\ +x_{2,j}r_j][(x_{1,l} + b_{\chi_1} - L_1)r_l - x_{2,l}q_l]\}, \\ m_i \equiv \|\mathbf{v}_i\| = \sqrt{(L_1 - b_{\chi_1} - x_{1,i})^2 + x_{2,i}^2}, \\ q_i \equiv \sqrt{m_i^2 - r_i^2}, \forall i \in I \end{cases} \quad (29)$$

Where, all of the above equations are subject to (14)-(15), which determine the values of $l, j \in I$.

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