49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Necessary Conditions for Optimality for a Distributed Optimal Control Problem

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Abstract— **This paper presents a novel optimal control problem formulation and new optimality conditions, referred to as distributed optimal control, for systems comprised of many dynamic agents that can each be described by the same ordinary differential equations (ODEs). The macroscopic system performance is represented by an integral cost function of a restriction operator comprised of the probability density function of the individual agents' state variables, and of their control laws. It is shown that, under proper assumptions, the macroscopic cost can be optimized subject to a hyperbolic partial differential equation (PDE) that describes the evolution of the macroscopic state over larger spatial and temporal scales. This methodology extends the capabilities of optimal control to complex systems described by numerous interacting dynamical systems. The approach is demonstrated on a simulated network of distributed sensors installed on autonomous underwater vehicles, and deployed to provide track coverage over a region of interest.**

I. Introduction

Traditionally, optimal control has dealt with a single dynamical system, such as a vehicle or chemical process, modeled by a small system of ordinary differential equations (ODEs). In many complex systems of current interest, ranging from ecology to materials science, and from chemistry to engineering, the goal is to optimally control numerous interacting dynamical systems or *agents* that can each be described by a detailed microscopic model in the form of an ODE. The macroscopic coherent behavior or coarse dynamics of these systems can typically be described by partial differential equations (PDEs), by deriving them from first principles, and by mapping the microscopic agents' states to a macroscopic description using an appropriate restriction operator that, in this paper, is assumed to be a probability density function (PDF). In the biological world and work related to swarm intelligence, intricate high-level system tasks are accomplished by solving a distributed optimization problem with many agents by adhering to a set of simple rules or control laws, such as when colonies of ants cooperatively forage for food [1]. However, there currently exist no methods for optimizing the macroscopic behavior of these systems, subject to microscopic agent dynamics.

In principle, optimal control can be extended to a set of dynamical systems by considering a set of coupled differential equations and by formulating their cooperative performance as an integral cost function of the combined state and control vectors. However, the computational complexity associated with solving the corresponding optimality conditions typically becomes prohibitive when the number of agents is very large. Therefore, despite its effectiveness at solving control and optimization problems in dynamic settings, the applicability of optimal control theory to distributed dynamical systems to date has been very limited. This paper presents a novel approach, referred to as distributed optimal control (DOC), which optimizes an integral function of the macroscopic system state, subject to microscopic dynamics and control laws. The necessary conditions for optimality for this novel DOC problem are derived in this paper, and demonstrated numerically through a mobile sensor network application.

The development of reliable sensor networks and autonomous-vehicle technologies are producing advanced surveillance systems that are characterized by a high degree of functionality and reconfigurability. Examples include mobile sensor networks for tracking and monitoring endangered species [2], or for tracking and detecting possible intruders [3]. Many of these applications employ wireless sensors that are installed on autonomous vehicles, and are deployed in large numbers to cooperatively detect, classify, localize, or track (DCLT) multiple targets in highly-variable and nonlinear environments. Several authors have addressed the placement of sensors to provide a desired quality of service assuming that they remain stationary [4], [5]. Approaches for generating a sensor trajectory include area coverage [6], [7], random [7], grid [8], and optimal search strategies [8], [9]. Cooperative control methods have been developed to provide area coverage [6], [10], or to cooperatively manage the sensors' formation in response to the sensed environment [7], [11]. Although optimal control is arguably the most general and effective approach to trajectory optimization [12], [13], its applicability to mobile sensor networks to date has been very limited [14] due to the lack of suitable DCLT objective functions, and to the computational complexity associated with solving the optimality conditions numerically for a large number of sensors.

Recently, DCLT objective functions for cooperative sensor networks have been obtained as a function of the sensors' probability distribution in the region of interest [3], [15]–[17] When the sensors are stationary, an approximately optimal sensors' distribution can be determined in the form of a parameterized Gaussian mixture by computing the mixing proportions via genetic algorithms [17]. However, no technique is presently available for optimizing DCLT objectives subject to the vehicles' nonlinear dynamics and to time-varying environmental conditions.

In this paper, the distributed optimal control approach

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is demonstrated by controlling a large sensor network of autonomous underwater vehicles (AUVs), where the network objective is to maximize the probability of cooperative track detection over a rectangular region of interest (ROI), and a fixed time interval. The remainder of the paper is organized as follows. The classical optimal control problem formulation, and numerical methods of solution are reviewed in Section II. The distributed optimal control problem and new optimality conditions are presented in Section III, and a numerical solution approach is proposed in Section IV. The application example involving a cooperative sensors network deployed to detect moving targets is described in Section V, and the numerical simulations results are presented in Section VI.

II. Background on Classical Optimal Control

Optimal control can be considered the most general approach to optimizing the performance of a dynamical system over time. Since its inception in the early 1970s, it has been applied to a variety of dynamical systems, including physical, chemical, economic, mechanical and air vehicles, in order to derive optimal control laws or trajectories. The classical optimal control formulation considers a system whose dynamics can be approximated by a small system of ODEs,

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t], \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

where, $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n$ is the system state, and $\mathbf{u} \in \mathcal{U} \subset \mathbb{R}^m$ is the control [12]. The dynamics in (1) also depend on system parameters that represent the physical characteristics of the system and scale the system's response to control inputs and to its own motions. Optimal control seeks to determine the state and control trajectories that optimize an integral cost function,

$$J = \phi[\mathbf{x}(t_f)] + \int_{t_0}^{t_f} \mathcal{L}[\mathbf{x}(t), \mathbf{u}(t), t]dt \tag{2}$$

over a time interval $[t_0,\ t_f]$, subject to (1) and, potentially, to an r-dimensional inequality constraint

$$\mathbf{q}[\mathbf{x}(t), \mathbf{u}(t), t] \leq \mathbf{0}_{r\times 1} \tag{3}$$

The necessary conditions for optimality are given by the well-known Euler-Lagrange equations, which can be derived using calculus of variations, as shown in [12]. When the system dynamics are linear and the cost function is quadratic, a linear-optimal control law known as linear quadratic regulator (LQR) can be obtained from the matrix Riccati equation with a terminal condition, and its solutions constitute necessary and sufficient conditions for optimality. For a nonlinear system and a general cost function, the necessary conditions for optimality amount to a Hamiltonian boundary-value (HPBV) problem for which there are no closed-form solutions and, therefore, they typically are solved numerically [12], [18]. As reviewed in [18], numerical methods for solving optimal control problems can be classified into direct and indirect methods. Indirect methods solve the HPBV problem numerically to determine candidate optimal trajectories known as extremals. Direct methods determine near-optimal solutions by discretizing the continuous-time problem about collocation points and then transcribing it into a finite-dimensional nonlinear program (NLP). The NLP is then solved using an appropriate optimization method, such as sequential quadratic programming (SQP) [19]. Direct methods are typically easier to implement than indirect methods and can be applied to a wider range of optimal control problems [20], [21].

If the observation process is uncertain or the dynamic system is forced by random disturbances, then the problem is referred to as a stochastic optimal control problem. In the optimal control literature to date, emphasis has been placed on the class of stochastic systems with small random effects because useful solutions are not yet available for the stochastic optimization of nonlinear systems with random variables of arbitrary probability distributions [12, pg. 421]. Furthermore, despite its effectiveness at solving control and optimization problems involving a single dynamical system, the applicability of optimal control to distributed dynamical systems to date has been very limited [14]. In principle, optimal control can be extended to N dynamical systems by considering N coupled differential equations and by formulating their cooperative performance as a single cost function of an Nn-dimensional state and an Nm-dimensional control, where n and m are the dimensions of the microscopic state and control, respectively. However, the computational complexity of solving the corresponding optimality conditions typically becomes prohibitive for large values of N. Additionally, the classical optimal control formulation is not well-suited to systems where the effects of random inputs are significant, and to systems in which the macroscopic dynamics cannot be derived in closed form.

III. Distributed Optimal Control

Although optimal control is considered the most general and effective approach to trajectory optimization [13], [22], so far its applications to large dynamical systems have been very limited due to the following limitations: exponential computational complexity of solutions with respect to the number of agents N; existing optimality conditions assume that random effects are small and can be described by additive normal distributions with zero mean; and the macroscopic analytical description of the system dynamics must be provided as a set of ordinary differential equations. In order to overcome these limitations, the proposed research considers a new optimal control problem formulation that optimizes the macroscopic performance of the multiscale dynamical system, which is obtained via an appropriate restriction operator and subject to microscopic agent dynamics and controls. In this paper, the approach is developed for the case in which the macroscopic description is given by the PDF of the microscopically evolving agent state, and the macroscopic evolution equation can be obtained in closed form from the continuity equation. Future work will consider

other restriction operators and macroscopic equations, such as those described in [23].

A. Distributed Optimal Control Problem Formulation and Assumptions

Consider the problem of optimally planning the trajectories of N cooperative, microscopic dynamical systems referred to as agents. Each agent may consist of an autonomous vehicle, sensor, or asset that can be described by a small system of ODEs,

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}[\mathbf{x}_i(t), \mathbf{u}_i(t), t], \quad \mathbf{x}_i(t_0) = \mathbf{x}_{i_0}, \tag{4}$$

referred to as a *detailed* equation. Its microscopic, detailed solution may be obtained numerically by a solution operator or time stepper $\mathcal{T}_d^t$ and, in the case of coupled agent dynamics or controls, must be obtained by solving a system of N coupled ODEs in the form (4), with $i = 1, \ldots, N$. It is assumed that all agents operate in a bounded subset of an Euclidean space, denoted by $\mathcal{A}$, and referred to as the operating region of interest (ROI), during a time interval $[T_0,\ T_f]$. A common feature of complex system is the emergence of macroscopic, coherent behavior from the interactions of microscopic agents. We assume that the macroscopic behavior can be derived from the microscopic one (i.e., the detailed agents' equations). Initially, the macroscopic description will be provided by a time-varying probability density function (PDF) $X(t) = \Phi_t(\mathbf{x}_i, t) \equiv \wp(\mathbf{x}_i, t)$, where $\wp : \mathcal{A} \times \mathbb{R} \to \mathbb{R}$. In other words, the i^{th} agent is described by a random vector with an arbitrary probability distribution, $\wp(\mathbf{x}_i, t)$, which provides the probability that the i^{th} agent has a state value $\mathbf{x}_i \in \mathcal{A}$, at any $t \in [T_0,\ T_f]$. Then the PDF $\wp(\mathbf{x}_i, t)$, referred to as *agent distribution*, can also be viewed as a description of the density or concentration of agents in space and time. The agent distribution provides a macroscopic description of the agents over time and space scales ($\mathcal{A}$ and $[T_0,\ T_f]$) that are much larger than the microscopic times and spatial domains ($\mathcal{X}$ and $[t_0,\ t_f]$).

Ultimately, we seek to optimize the macroscopic performance of the distributed system, represented as an integral function of the agent distribution and control,

$$J = \phi[\wp(\mathbf{x}_i(T_f), T_f)] + \int_{T_0}^{T_f} \int_{\mathcal{A}} \mathcal{L}[\wp(\mathbf{x}_i, t), \mathbf{u}(t), t]\ d\mathbf{x}_i dt \tag{5}$$

subject to the evolution of $\wp(\mathbf{x}_i, t)$. If every agent can be modeled by the detailed equation in (4), the macroscopic evolution equation can be derived by considering an infinitesimal control volume in $\mathcal{A}$ and by assuming that all agents remain in $\mathcal{A}$ at all times and are neither created or destroyed. In this case, the time-rate of change of the distribution $\wp(\mathbf{x}_i, t)$ is given by the advection equation, which governs the motion of a conserved scalar quantity as it is advected by a known velocity field [24],

$$\begin{aligned} \frac{\partial \wp}{\partial t} &= -\nabla \cdot [\wp(\mathbf{x}_i, t)\ \dot{\mathbf{x}}_i] \\ &= -\nabla \cdot \{\wp(\mathbf{x}_i, t)\ \mathbf{f}[\mathbf{x}_i(t), \mathbf{u}_i(t), t]\} \end{aligned} \tag{6}$$

where $\nabla \cdot \mathbf{v}$ denotes the divergence of a vector field $\mathbf{v} \in \Re^n$. Therefore, the macroscopic evolution equation is a hyperbolic PDE that provides the dynamic constraints for the optimization of the cost function (5). The initial and boundary conditions for (6), as well as an admissability constraint (A.C.), are given by the initial agent distribution, $\wp_0(\mathbf{x}_i)$, and by the normalization condition that must be satisfied by the PDF at any time thereafter, i.e.:

$$\text{I.C.:} \quad \wp(\mathbf{x}_i, T_0) = \wp_0(\mathbf{x}_i); \tag{7}$$

$$\text{B.C.:} \quad \wp(\mathbf{x}_i \in \partial\mathcal{A}, t) = 0, \quad \forall t \in [T_0, T_f]; \tag{8}$$

$$\text{A.C.:} \quad \int_{\mathcal{A}} \wp(\mathbf{x}_i, t) d\mathbf{x}_i = 1; \tag{9}$$

$$\wp(\mathbf{x}_i \notin \mathcal{A}, t) = 0, \quad \forall t \in [T_0, T_f] \tag{10}$$

The initial conditions are given by the initial agent distribution, and the admissability constraint is provided by the normalization condition implicit in the PDF.

The goal of distributed optimal control is to determine the agent distribution, $\wp^*(\mathbf{x}_i, t)$, and state-feedback control law, $\mathbf{u}^*(t)$, that optimize the macroscopic performance (5), subject to (6) and (7)-(10). It can be seen that the above problem formulation violates the classical optimal control problem formulation and, therefore, existing optimality conditions [12] cannot be utilized to determine $\wp^*(\mathbf{x}_i, t)$ and $\mathbf{u}_i^*(t)$. An example of an evolving PDF subject to the described conditions is illustrated in Figure 1, where an initial distribution is given and the optimal control law is assumed to "move" the distribution in a circle about the center of the spatial domain at a constant speed, which causes a spiraling effect. As shown in the next subsection, the calculus of variations may be used to derive a new set of optimality conditions that amount to a set of PDEs that can be solved numerically, as shown in Section IV.

B. Optimality Conditions

Optimal solutions for the optimal control of distributions are defined by equations that establish necessary and sufficient conditions for minimum cost. According to the calculus of variations approach [12], necessary conditions must be satisfied in order for the cost function to be *stationary*. If the necessary conditions derived in this section are satisfied, *sufficient conditions* guarantee optimality of an extremal because they establish whether the extremal minimizes or maximizes the cost function. Once stationary of the cost function is established, however, the higher-order sensitivity to control variations can be used to satisfy the sufficient conditions for optimality. Since the evolution equation (6) must be satisfied over the entire time interval $[T_0,\ T_f]$, it can be written as the following equality constraint using the product rule,

$$\frac{\partial \wp}{\partial t} + (\nabla \wp) \cdot \mathbf{f} + \wp(\nabla \cdot \mathbf{f}) = 0 \tag{11}$$

where the functions' arguments are omitted for brevity. Then, similarly to static optimization, the above equality constraint is adjoined to the integrand of J using a time-varying

Lagrange multiplier λ, to obtain the augmented cost function

$$\begin{aligned} J_A &= \phi[\wp(\mathbf{x}_i(T_f), T_f)] + \int_{T_0}^{T_f}\int_{\mathcal{A}} \{\mathcal{L}(\wp, \mathbf{u}_i, t) \\ &+ \lambda\left[\frac{\partial \wp}{\partial t} + (\nabla\wp)\cdot\mathbf{f} + \wp(\nabla\cdot\mathbf{f})\right]\Big\}\ d\mathbf{x}_i dt \\ &= \phi[\cdot] + \int_{T_0}^{T_f}\int_{\mathcal{A}} \Big\{\mathcal{L}(\wp, \mathbf{u}_i, t) + \lambda\ \wp(\nabla\cdot\mathbf{f}) + \lambda\ \frac{\partial\wp}{\partial t} \\ &+ \lambda\ (\nabla\wp)\cdot\mathbf{f}\}\ d\mathbf{x}_i dt \end{aligned} \tag{12}$$

The notation $\nabla\wp = (\partial\wp/\partial\mathbf{x}_i)^T$ is a row vector of partial derivatives or *gradient* of $\wp$, and $(\cdot)$ denotes the dot product. According to the calculus of variations, the integrand of (12) must satisfy stationarity conditions throughout $[T_0,\ T_f]$ in order for J_A to be stationary. The following Hamiltonian is introduced,

$$\mathcal{H} \equiv \mathcal{L}(\cdot) + \lambda\ \wp(\nabla\cdot\mathbf{f}) = \mathcal{H}[\wp(\mathbf{x}_i, t), \mathbf{u}_i(t), \lambda(t), t] \tag{13}$$

which is a function of the agent distribution, the control, and the Lagrange multiplier. Subsequently, the augmented cost function (12) can be simplified as follows,

$$\begin{aligned} J_A &= \phi[\cdot] + \int_{T_0}^{T_f}\int_{\mathcal{A}} \{\mathcal{H}[\wp(\mathbf{x}_i, t), \mathbf{u}_i(t), \lambda(t), t] \\ &+ \lambda(t)\ \frac{\partial\wp(\mathbf{x}_i, t)}{\partial t} + \lambda(t)\ \nabla\wp(\mathbf{x}_i, t)\ \mathbf{f}[\cdot]\Big\}\ d\mathbf{x}_i dt \\ &= \phi[\cdot] + \int_{T_0}^{T_f}\int_{\mathcal{A}} \mathcal{H}[\cdot]\ d\mathbf{x}_i dt + (\lambda\ \nabla\wp\ \mathbf{x}_i)|_{t=T_f} \\ &- (\lambda\ \nabla\wp\ \mathbf{x}_i)|_{t=T_0} + \int_{T_0}^{T_f}\int_{\mathcal{A}} \frac{d}{dt}\left[\lambda\ \nabla\wp\right]\mathbf{x}_i\ d\mathbf{x}_i dt \end{aligned} \tag{14}$$

by using integration by parts and by noting that $\int_{\mathcal{A}} \partial\wp/\partial t\ d\mathbf{x}_i = 0$ from (9) and Leibniz integral rule.

For stationarity, the first-order effect of control variations $\delta\mathbf{u}_i$ on the cost function must be zero throughout $[T_0,\ T_f]$. Because control variations generally lead to subsequent state perturbations due to the causality of the dynamic equation (4), the first variation of the augmented cost function is given by,

$$\begin{aligned} \delta J_A &= \int_{T_0}^{T_f}\int_{\mathcal{A}} \Big\{\frac{\partial\mathcal{H}[\cdot]}{\partial\mathbf{x}_i}\delta\mathbf{x}_i(\delta\mathbf{u}_i) + \frac{\partial\mathcal{H}[\cdot]}{\partial\mathbf{u}_i}\delta\mathbf{u}_i \\ &- \frac{d}{dt}\left[\lambda\ \nabla\wp\right]\delta\mathbf{x}_i(\delta\mathbf{u}_i)\Big\}\ d\mathbf{x}_i dt \\ &+ \left.\frac{\partial\phi[\cdot]}{\partial\mathbf{x}_i}\right|_{t=T_f}\delta\mathbf{x}_i(\delta\mathbf{u}_i) \\ &+ (\lambda\ \nabla\wp\ \mathbf{x}_i)|_{t=T_f}\,\delta\mathbf{x}_i(\delta\mathbf{u}_i) \\ &- (\lambda\ \nabla\wp\ \mathbf{x}_i)|_{t=T_0}\,\delta\mathbf{x}_i(\delta\mathbf{u}_i) \end{aligned} \tag{15}$$

where $\mathbf{x}_i(\delta\mathbf{u}_i)$ denotes state variations arising from control perturbations. It can be assumed that $\delta\mathbf{u}_i$ have no effects on the initial state. For an extremum, we must have $\delta J_A = 0$ for all $\delta\mathbf{x}_i, \delta\mathbf{u}_i$, and each part of δJ_A must equal zero separately near the optimal solution. Thus, it can be shown that the equations,

$$\begin{aligned} \frac{\partial\mathcal{H}[\cdot]}{\partial\mathbf{x}_i} &= \mathbf{0} \quad \text{or} \\ \dot{\lambda}(t)\nabla\wp &= \frac{\partial\mathcal{L}[\cdot]}{\partial\mathbf{x}_i} + \lambda(t)\left\{\nabla\wp(\nabla\cdot\mathbf{f})\right. \\ &+ \left.\wp[\nabla\ \mathbf{F}]^T - \nabla^2\wp\ \mathbf{f}[\cdot] - \frac{\partial}{\partial t}(\nabla\wp)\right\} \end{aligned} \tag{16}$$

and,

$$\frac{\partial\mathcal{H}[\cdot]}{\partial\mathbf{u}_i} = \mathbf{0} \quad \text{or} \quad \frac{\partial\mathcal{L}[\cdot]}{\partial\mathbf{u}_i} + \lambda(t)\ \wp\ [\nabla\mathbf{G}]^T = \mathbf{0} \tag{17}$$

must be satisfied for $T_0 \le t \le T_f$, subject to the terminal conditions

$$\lambda(T_f)\ \nabla\wp(\mathbf{x}_i(T_f), T_f) = -\nabla\phi|_{t=T_f} \tag{18}$$

where $\mathbf{F} \equiv \partial\mathbf{f}/\partial\mathbf{x}_i$ and $\mathbf{G} \equiv \partial\mathbf{f}/\partial\mathbf{u}_i$ denote Jacobian matrices obtained from the linearized dynamics. Additionally, the optimal distribution must satisfy the initial, boundary, and admissibility conditions in (7)-(10). Since the inputs, $\mathbf{u}_i$, to the individual agents are controllable, it is reasonable to assume that in most applications they are a deterministic function of the state as in classical optimal control, i.e., $\mathbf{u}_i(t) = \mathbf{c}[\mathbf{x}_i(t), t]$, where $\mathbf{c} : \mathcal{X}\times\mathbb{R}\to\mathcal{U}$ is the control law. Then, the stochastic optimal control of distributions amounts to determining the optimal agent distribution $\wp^*(\cdot)$ and optimal control law $\mathbf{c}^*[\cdot]$ that satisfy the optimality conditions (7)-(10),(16)-(18). If these necessary conditions are met and any control perturbations increase the cost, then the solution can be considered a minimum-cost extremal. Otherwise, proving that second or higher-order derivatives of the Hamiltonian in (13) with respect to the control are positive definite can be used as a sufficient convexity condition.

IV. Approximate Numerical Solution via Nonlinear Programming

The stochastic optimal control of distributions problem presented is solved numerically to maximize the probability of track detection for the mobile sensor network, where the direct approach used implements finite element and finite difference methods. An approximation of the optimal sensor distribution can be found in the form of a time-varying Gaussian mixture with mixing proportions, $w_1, ..., w_z$, and component densities $f_1(x_i, t), ..., f_z(x_i, t)$. The individual component densities are also time-varying PDFs and are characterized by their means, $\mu_j(t)$, and standard deviations, $\sigma_j(t)$. Each density is completely within $\mathcal{A}$ and zero elsewhere and on ∂A. Then (29),(6), and (7) are calculated at discrete points in state space and time, where $\wp(\mathbf{x}_i(t_k), t_k) = \sum_j w_j(t_k) f_j(\mathbf{x}_i(t_k), t_k) = \wp_k$ and $t_k = k\Delta t,\ k = 1, \ldots, K$, with Δx_l and Δt as the

discretization intervals. At each t_k, (29) is evaluated using a multi-dimensional fast Fourier transform (FFT) algorithm, as is demonstrated in [25]. The values between the collocation points are taken to be piecewise-constant. The evolution equation (6) is solved by finite difference, where ρ_k is defined as the finite difference approximation of the PDE operator, such that $\wp_{k+1} = \wp_k + \Delta t \rho_k$. The problem can then be converted into a finite-dimensional nonlinear program (NLP):

$$\text{minimize } J(\boldsymbol{\chi}) = \Delta t \sum_{l=1}^{n} \Delta x_l \sum_{k=1}^{K} \sum_{\mathbf{x}_k \in \mathcal{A}} \mathcal{L}[\wp_k, \mathbf{u}_k, t_k], \quad (19)$$

$$\text{subject to } \wp_{k+1} - \wp_k - \Delta t \rho_k(\boldsymbol{\chi}) = 0, \quad k = 0, \ldots, K-1 \quad (20)$$

$$\sum_{l=1}^{n} \Delta x_l \sum_{\mathbf{x}_k \in \mathcal{A}} \wp(\mathbf{x}_k, t_k) - 1 = 0, \ k = 0, \ldots, K-1 \quad (21)$$

where the variables are the controls and the parameters of the Gaussian mixture, and the initial and boundary conditions can be affixed as equality constraints. The NLP gives a solution, χ^*, that characterizes an approximation of the optimal distribution, $\wp^*(x_i, t_k)$, and control history $u_i^*(t_k)$.

The optimal macroscopic description can be used determine the microscopic control laws by utilizing a potential field method. The potential field method is a well-known motion planning approach that treats a robot as a particle under the influence of an artificial potential function, U. Several potential field methods have been developed for generating a collision-free path for a single mobile robot that must travel from an initial configuration to a goal configuration [26], [27]. In these methods, the potential function is the sum of an attractive potential U_{att} that "pulls" the robot toward a goal state $\mathbf{x}_f$ and a repulsive potential U_{rep} that "pushes" the robot away from the obstacles. After U is defined, the method is implemented by discretizing the state space $\mathcal{A}$, and by evaluating the potential function for all discrete values of $\mathbf{x}_i$ in $\mathcal{A}$ using a finite resolution grid. A novel potential field method has been developed in order to determine *microscopic* control laws for the individual agents, such that they are "pulled" toward the optimal PDF, and achieve the desired macroscopic behavior. The potential function is defined as a linear combination of an attractive potential, representing the desired distribution (PDF) of the agents, and a repulsive potential, representing collision avoidance between agents and with obstacles. When $\wp^*_{\mathbf{x}}(\mathbf{x}_i, t_k)$ is integrated over a region $\mathcal{R} \subset \mathcal{A}$, it provides the probability that the i^{th} agent is located in $\mathcal{R}$ at t_k in an optimal distribution of agents, i.e., the probability mass $\Pr(\mathbf{x}_i \in \mathcal{R}, t_k) = \int_{\mathcal{R}} \wp^*(\mathbf{x}_i, t_k) d\mathbf{x}_i$. In order to downgrade the probability mass based on an actual agent's state, the attractive potential is generated using a desired posterior PDF defined as,

$$U_{att}(\mathbf{x}_i, t_k) = -\wp^*(\mathbf{x}_i, t_k) \ L[\mathbf{x}_i, t_k \mid X(t_{k-1})] \quad (22)$$

where $L[\cdot]$ is the likelihood of state $\mathbf{x}_i$ at t_k, given the macroscopic state at the previous time step, $X(t_{k-1})$, based on the actual agent distribution at t_{k-1}. The repulsive potential U_{rep} is defined as an exponential function of the shortest distance between the agents and the obstacles, as in [26], [27]. Then, a stabilizing feedback control law for tracking the optimal control history, $\mathbf{u}_i^*(t_k)$, can be obtained from the negative gradient of the potential function. When the restriction operator Φ_t is not a PDF, an appropriate lifting operator μ can be used to generate the attractive potential by mapping the optimal macroscopic description $\mathbf{X}^*(t)$ to N consistent and optimal microscopic descriptions, $\mathbf{x}_1^*(t), \ldots, \mathbf{x}_N^*(t)$.

V. Distributed Optimal Control Application to Cooperative Sensor Networks

The novel distributed optimal control methodology discussed in the previous sections is demonstrated on a cooperative track detection problem in sensor networks, where the fundamental objective is to provide an optimal probability that a target track through a rectangular region of interest (ROI) will be detected by various independent sensors at several times. Cooperative track detection is known to be well-suited to systems where no knowledge about the targets is given *a priori*, and the sensors are likely to report false alarms while opportunities for correct target detections are infrequent. Cooperative track detection fuses multiple closest-point-of-approach (CPA) detections from different sensors to confirm detections, and the tracks of an unknown number of targets can be constructed from multiple consecutive frames of observations provided by low-cost sensors utilizing multiple hypothesis tracking (MHT) [28] or geometric invariants [29] algorithms. However, there are no current methods capable of optimizing the objectives of large sensor networks subject to nonlinear vehicular dynamics and time-varying environmental states. It is shown in this paper that distributed optimal control is effective even with such difficult conditions.

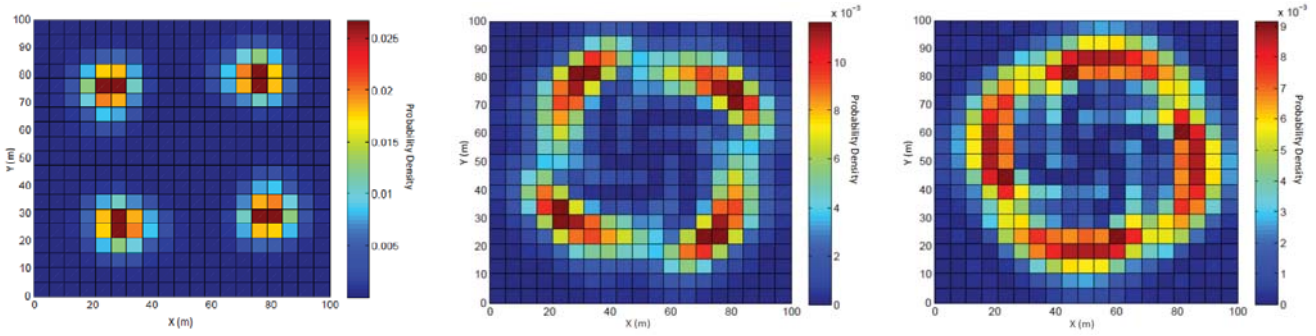
The network of omnidirectional sensors is deployed in a ROI, $\mathcal{A} = [0, L] \times [0, L] \subset \Re^2$, which may be occupied by B convex obstacles $\{\mathcal{B}_1, \mathcal{B}_2\} \subset \mathcal{A}$. The obstacle locations are not known *a priori*, but they may be detected at any time $t \in [T_0, T_f]$. Each sensor is mounted on an autonomous underwater vehicle (AUV) with the dynamic model,

$$M(\mathbf{x}_j)\ddot{\mathbf{x}}_j + h(\mathbf{x}_j, \dot{\mathbf{x}}_j) + g(\mathbf{x}_j) = u(\mathbf{x}_j), \ j = 1, \ldots, m \quad (23)$$

where $M(\mathbf{x}_j)$ is the vehicle's inertia matrix, $h(\mathbf{x}_j, \dot{\mathbf{x}}_j)$ is the fictitious force, $g(\mathbf{x}_j)$ is the gravitational force, and $u(\mathbf{x}_j)$ is the torque input [30]. The sensors receive isotropic energy which is weakened by the environment in relation to the power law,

$$E_j(t) = cF[\lambda_j(t)]^{-\alpha} \quad (24)$$

where $\lambda_j(t)$ is the distance between the location of the jth sensor, $\mathbf{x}_j$, and the target position at time t. The attenuation coefficient, α, and the scaling constant, c, are chosen based on the environmental conditions and the mechanisms governing wave propagation. In this paper, both are assumed to be known and constant. The target source level, F, is assumed to be constant and independent of the target location



(a) (b) (c)

Fig. 1. Example of time-varying PDF evolved by the advection equation (6) at times $t = 0$ hr (a), $t = 4$ hr (b), and $t = 6$ hr (c).

and time. A CPA detection then occurs when E_j surpasses a user-selected threshold value, ϑ_j. At the time of a CPA detection event, the jth sensor transmits the values of E_j and $\mathbf{x}_j$ to the central processor. Because the received signal decays over distance, as governed by (24), the greatest range from which a CPA detection can be made is,

$$r_j = (cF)^{1/\alpha}\vartheta_j \tag{25}$$

If the effect of the vehicle propulsion on sensing is assumed to be negligible, r_j can be estimated based on the existing environmental conditions. Therefore, it can assumed to be known and constant for all sensors $j = 1, ..., N$, and a CPA may only occur if the target travels within the range of r_j from sensor j. To form an estimated target track, a minimum of γ elementary detections are needed from γ distinct sensors within the network. A value of $\gamma = 3$ was found to give accurate tracking by proximity sensors subject to few false alarms, and errors normally distributed with a standard deviation of 20% [15].

Since the AUVs use omnidirectional sensors that follow the isotropic law (24), the field-of-view (FOV) of a sensor, denoted by $C_j(t) = C_j[\mathbf{x}_j(t), r_j]$, at time t can be defined as a disk of constant radius r_j and centered at the sensor position, $\mathbf{x}_j$. The sensor can then be viewed as a disk, illustrated in Figure 2 that moves in $\mathcal{A}$ consistent with (23). Let the FOVs of the full network of sensors be represented by the set $S(t) = \{C_1(t), ..., C_N(t)\}$, which is characterized by the sensor ranges, $r_1(t), ..., r_N(t)$, and the AUV positions, $\mathbf{x}_1(t), ..., \mathbf{x}_N(t)$.

The sensors' states and the targets' speeds, headings, and initial positions are regarded as random variables governed by the joint PDFs $\wp(\mathbf{x}_j, t)$, $f_V(V, t)$, $f_\theta(\theta, t)$, and $f_T(\mathbf{x}_{T_0}, t)$, respectively. The PDF of the sensors' states is a function of time since the sensors move to optimize the network's track coverage, and the PDFs of the target tracks' variables are assumed to be known functions of time computed using the tracking methods in [15]. Then the detection region, $\Omega_T \subset \mathcal{A}$ is expanded isotropically from the target track, given by

$$\mathbf{x}_T(t) = \mathbf{x}_{T_0} + V[\cos\theta \;\; \sin\theta]^T dt \tag{26}$$

over a time differential $dt \subset [T_0,\ T_f]$, where $\mathbf{x}_T(t_0) = \mathbf{x}_{T_0} \in \mathcal{A}$. Suppose the event $D_j = \{0,\ 1\}$ corresponds to all

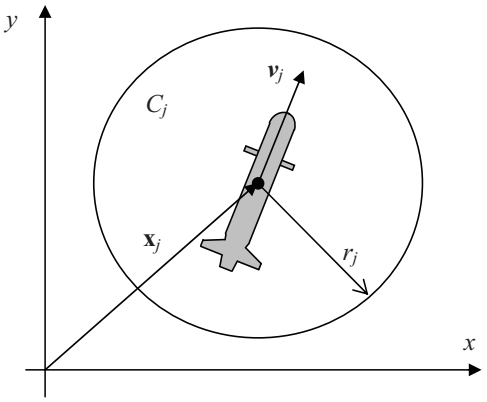


Fig. 2. Schematic of jth mobile sensor (not to scale) taken from [14].

possible mutually exclusive outcomes of sensor j, where if $D_j = 1$, the sensor is reporting a detection, and if $D_j = 0$, the sensor is not reporting. If the targets are assumed to be uniformly distributed in $\mathcal{A}$, the probability of sensor j reporting a detection is given by the spatial Poisson process,

$$\Pr\{D_j = 1 \mid \mathbf{x}_T(t) \in \mathcal{A}\} = 1 - e^{-\phi_t} \tag{27}$$

where,

$$\phi_t(\mathbf{x}_{T_0}, V, \theta) = \int_{T_0}^{T_f} \int_{\Omega_T(\mathbf{x}_{T_0}, \theta, V\,dt)} \wp(\mathbf{x}_j, t) d\mathbf{x} dt \tag{28}$$

is the coverage factor for a sensor sampled from $\wp(\mathbf{x}_j, t)$ with a detection region Ω_T. The coverage factor of a spatial Poisson process is defined as the expected number of points that occur in a subset of a Euclidean space, where every point that falls within this region results in the occurrence of a detection event, $D_j = 1$, for sensor j.

A central processor receives reports of detection events $\{D_1, ..., D_N\}$ from a network of N sensors, and it attempts to construct a target track from the incoming data. When $\sum_{j=1}^{N} D_j \geq \gamma$ is satisfied, a successful track detection is declared. Therefore the probability of a successful track detection by at least γ sensors can be expressed by using Bernoulli trials [3]. Assuming that individual detection events are independent and statistically identical, and that $\phi_t << 1$

and $N >> 1$, the probability of successful track detection in the ROI, $\mathcal{A}$, can then be approximated by an integral function of the sensors' PDF,

$$\begin{aligned} P_t &\equiv \Pr(\sum_{j=1}^{N} D_j \geq \gamma \mid \mathbf{x}_T(t) \in \mathcal{A}) \qquad (29) \\ &\approx 1 - \int_{T_0}^{T_f} \int_{0}^{2\pi} \int_{V_{min}}^{V_{max}} \int_{\mathcal{A}} e^{-N\phi_t(\mathbf{x}_{T_0},V,\theta)} f_T(\mathbf{x}_{T_0},t) \\ &\times \quad f_V(V,t) f_\theta(\theta,t) \sum_{z=0}^{\gamma-1} \frac{[N\phi_t(\mathbf{x}_{T_0},V,\theta)]^z}{z!} d\mathbf{x}_{T_0} dV d\theta dt \end{aligned}$$

as shown in [15], where V_{min} and V_{max} are the minimum and maximum speeds of the targets, and $\phi_t(\mathbf{x}_{T_0}, V, \theta)$ is a function of $\wp(\mathbf{x}_j, t)$, as seen in (28). This objective function is in the form of (5) and can be rewritten as the NLP (19) and solved as a time-varying Gaussian mixture, as described in Section IV. Then, the microscopic control law defined by the potential field method dictates the optimal paths to the individual sensors, which is also explained in Section IV.

VI. Simulations and Results

The DOC methodology presented in the previous sections is demonstrated here on a relatively large simulated ocean sensor network comprised of $N = 50$ sensors deployed at random positions in a ROI, $\mathcal{A} = [0,\ L_1] \times [0,\ L_2]$ where $L_1 = 90km$ and $L_2 = 82.51\ km$. The simulation was run over the time interval $t \in [T_0, T_f]$ where $T_0 = 0$ and $T_f = 12\ hr$. The sensor positions were updated at intervals of $t_s = 10sec$, and the optimal distribution was evolved every $\Delta t = 10min$ using the finite-dimensional NLP (19). The joint PDFs $f_v(V,t)$ and $f_T(\mathbf{x}_{T_0}, t)$, which represent the targets' speeds and initial positions, are assumed to be uniform and constant, but the PDF corresponding to the targets' headings was set arbitrarily and varies randomly over the search time duration. The sensor ranges for all vehicles are set as $r = 4\ km$, and the parameters defining the repulsive potential were chosen to avoid vehicle collisions and prevent sensor overlap. When a target travels within a sensor's range, the probability of detection is $P_D = 1$, and a successful track detection is declared when a target is detected by at least $\gamma = 3$ separate sensors. The target source amplitude is assumed to be constant

At each update, an approximate optimal sensor distribution is found numerically as a Gaussian mixture with $z = 20$ components, as described in Section IV. For computational simplicity, the standard deviation σ_j of each density j is held constant, but the means μ_j and mixing proportions w_j are solved to construct the distribution. At the start of the search period, the means and mixing proportions are given random initial values. The sensor network follows the PDF to adjust towards its optimal distribution by utilizing the potential field method explained previously. The sensor positions at several times during the simulation are plotted in Figure (3) with the PDF in the background. The distribution changes to maximize detection probability over the search time, and the sensors are seen to conform to the optimal PDF.

The performance of the network is measured with the search objective function from Equation (29), where the network goal is to maximize this value. Table I illustrates the improvements that are achieved using the distributed optimal control approach. The optimally controlled network is compared to cases where the PDF is held constant at the random initial value, as seen in Figure 3, and where the distribution is defined by a constant Gaussian mixture with $z = 20$ components and uniform mixing proportions spaced evenly in a rectangular grid formation across the ROI. While the random distribution is an unintelligent approach, a network with a uniform distribution is a reasonable strategy and is treated as a baseline case. As the evenly spaced distribution performs better than the random case, the optimally controlled method results in a performance that is 104.9% better than the random distribution and improved 46.0% compared to the uniform case.

TABLE I
Performance Comparison

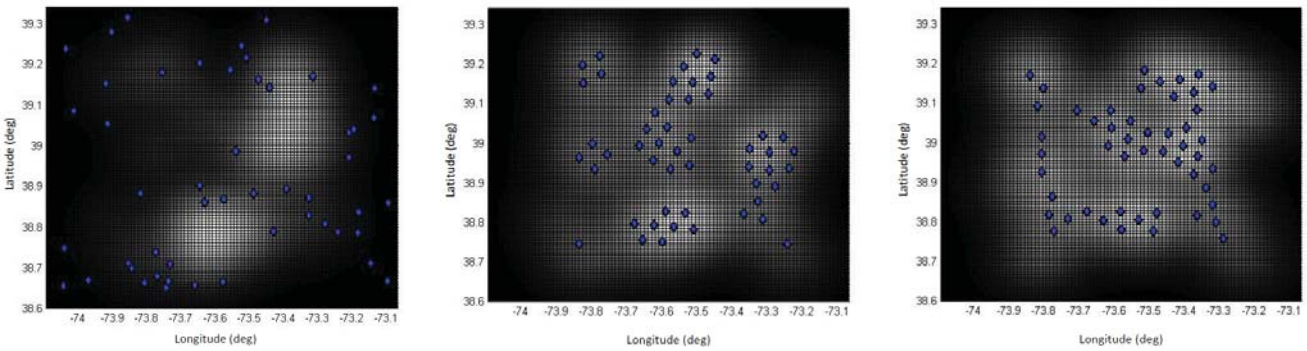
Example Case	Successful Detection Probability, P_t
Random Distribution	0.2122
Uniform Distribution	0.2978
Distributed Optimal Control	0.4348

In this simulation, the sensors are initially deployed at random locations and subsequently reconfigure based on the PDF by means of the potential field method. Another possibility is to first place the sensors at a set of N initial conditions sampled from an optimized initial PDF, such that the desired density is followed at all times. The former approach is used in this paper to illustrate the adaptive behavior of the method presented.

VII. Conclusion

This paper presents a novel optimal control problem formulation and derives novel optimality conditions for a class of problems referred to as distributed optimal control. The approach presented in this paper seeks to extend optimal control to a class of complex systems comprised of many dynamical systems that can each be described an ordinary differential equation, referred to as detailed microscopic model. Over large spatial and time scales, the interactions of these dynamical systems, or agents, give rise to macroscopic coherent behavior that can be described by a macroscopic state obtained via a restriction operator, such as a probability density function or higher moments of the agents distribution.

Assuming the macroscopic system state can be described by the time-varying probability density function of the agents' states, and the cost is represented by an integral function of the macroscopic state PDF, the distributed optimal control solution can be obtained by optimizing the cost function subject to a hyperbolic partial differential equation known as advection equation. The optimal probability density function can then be utilized to derive microscopic



(a) (b) (c)

Fig. 3. Sensor network positions (blue diamonds) and optimal sensor distributions at (a) $t = 0$ with randomly defined sensor positions and PDF, (b) $t = 6$ hr, (c) $t = 12$ hr.

control laws by means of a potential navigation function. The effectiveness of the approach is demonstrated on a distributed mobile sensor network of AUVs deployed in a region of interest near the coast of New Jersey to cooperatively track moving targets or intruders with no prior information. The numerical results show that the sensor network performance is improved by up to 104.9% compared to random or uniform sensor distributions.

ACKNOWLEDGMENTS

The authors would like to thank Prof. Yannis Kevrekidis at Princeton University, Princeton (NJ), and Dr. T. A. Wettergren at the Naval Undersea Warfare Center, Newport (RI), for their helpful guidance and suggestions.

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