Optimality & Uncertainty

TIME-DEPENDENT SURVEILLANCE-EVASION GAMES

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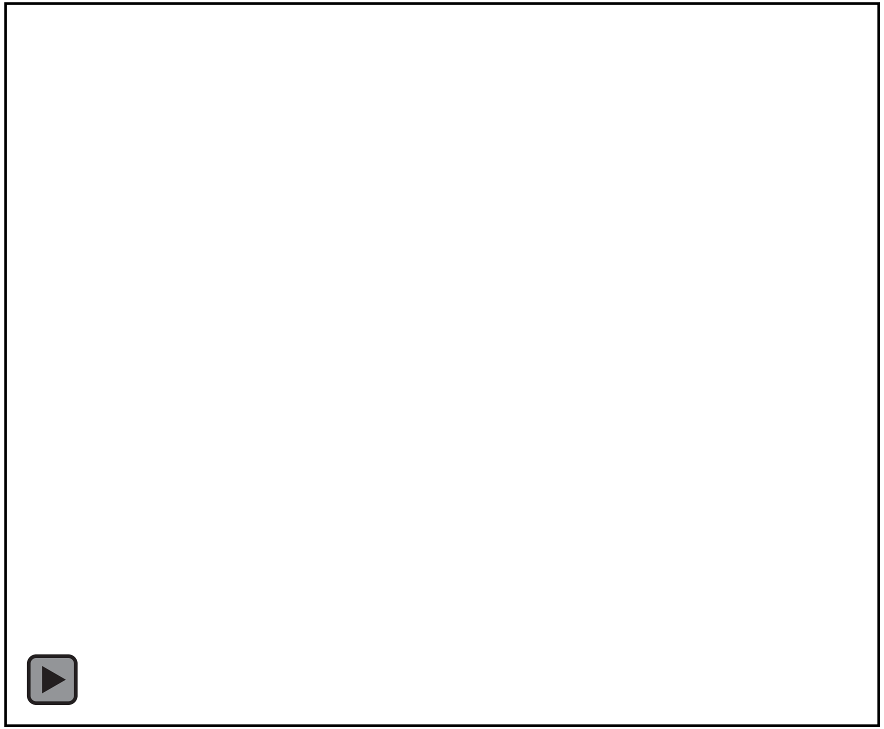
DETERMINISTIC TIME-DEPENDENT PROBLEM

A time-dependent pointwise observability function is defined to reflect the observer's surveillance capabilities for different parts of the domain.



Pointwise Observability:

$$K(\mathbf{x},t) = \begin{cases} \frac{1}{\|\mathbf{x}-\mathbf{x}_o\|^2+\epsilon} + \epsilon & \mathbf{x} \text{ is visible} \\ \epsilon & \mathbf{x} \text{ in shadow} \end{cases}$$

DETERMINISTIC TIME-DEPENDENT PROBLEM

The goal is to guide the evader from its source to its desired target while minimizing the cumulative observability along the way.

Cumulative Observability:

$$\mathcal{J}(\mathbf{x}, \mathbf{a}) = \int_0^{T_{\mathbf{a}}} K(\mathbf{y}(s), s)ds$$

Value function:

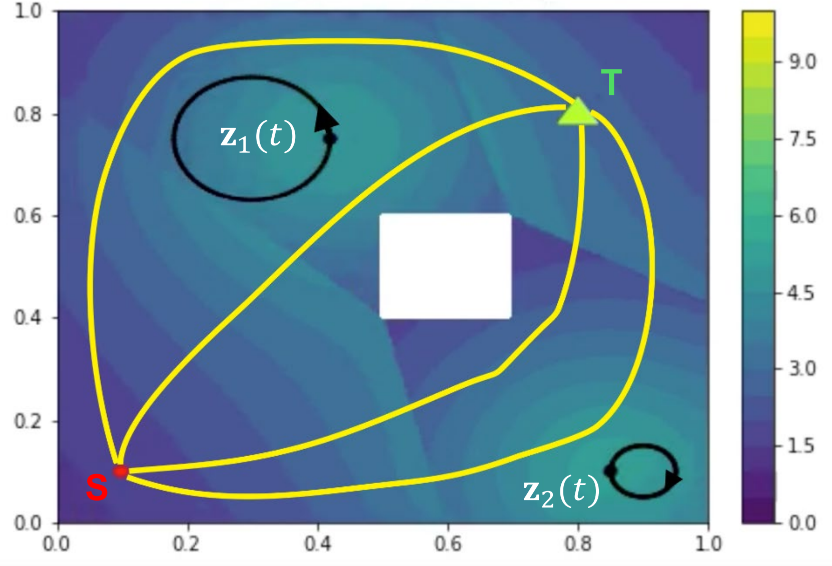
$$u(\mathbf{x}) = \inf_{\mathbf{a}(\cdot)} \mathcal{J}(\mathbf{x}, \mathbf{a})$$

$$\boxed{\begin{gathered} u_t(\mathbf{x}, t) - |\nabla u(\mathbf{x}, t)| f(\mathbf{x}) + K(\mathbf{x}, t) = 0 \\ u(\mathbf{x_T}, t) = 0, \quad \forall t \geq 0 \in \mathbb{R} \end{gathered}}$$

STRATEGIC GAME FORMULATION

Both the evader and the observer are required to choose a plan in advance, trying to anticipate the opponent's actions.

- Evader: $\boldsymbol{\theta}$, probability distribution over all (infinitely many) paths ($\boldsymbol{a}(\cdot)$) from S to T;
- Observer: $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, probability of patrol trajectories $Z = \{\mathbf{z}_1(t), \mathbf{z}_2(t)\}$;



Expected Observability along $\boldsymbol{a}(\cdot)$ Given $\boldsymbol{\lambda}$:

$$\mathcal{J}^{\lambda}(\mathbf{a}(\cdot)) = \lambda_1 \mathcal{J}_1(\mathbf{a}(\cdot)) + \lambda_2 \mathcal{J}_2(\mathbf{a}(\cdot))$$

Expected Observability with $\boldsymbol{\theta}$ Given $\boldsymbol{\lambda}$:

$$\mathbb{E}_{\theta}[\mathcal{J}^{\lambda}(\mathbf{a})]$$

STRATEGIC GAME FORMULATION

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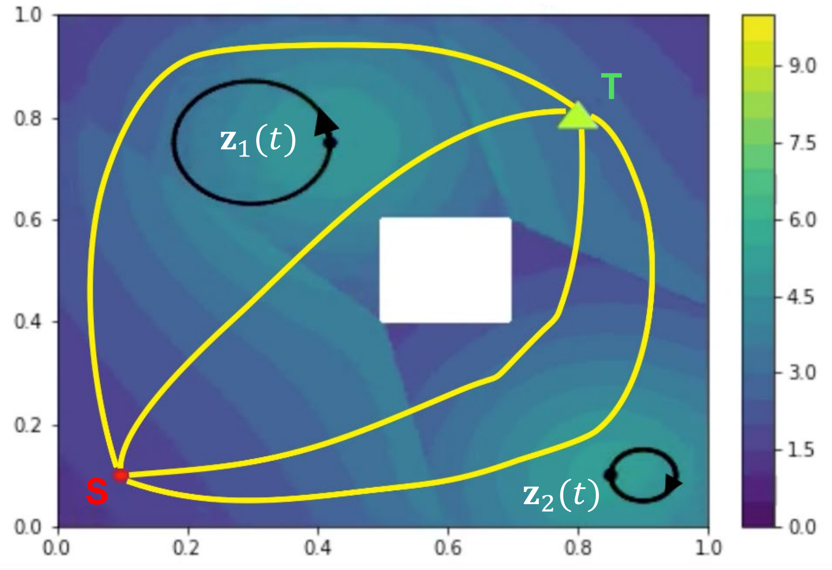


Evader: $\min_{\theta} \mathbb{E}_{\theta}[\mathcal{J}^{\lambda}(\mathbf{a})]$

Observer: $\max_{\lambda} \mathbb{E}_{\theta}[\mathcal{J}^{\lambda}(\mathbf{a})]$

Semi-infinite Zero-sum Game!

NASH EQUILIBRIUM

A mixed Nash equilibrium of a zero-sum game always exists, and is attained at the minimax (maximin).

- A pair of strategies (λ^*,θ^*) is a Nash equilibrium if both the observer and the evader are happy with their current strategies:

$$\mathbb{E}_{\theta*}[\mathcal{J}^{\lambda^*}(\mathbf{a}(\cdot))] \geq \mathbb{E}_{\theta*}[\mathcal{J}^{\lambda}(\mathbf{a}(\cdot))], \qquad \forall\lambda$$

$$\mathbb{E}_{\theta*}[\mathcal{J}^{\lambda^*}(\mathbf{a}(\cdot))] \leq \mathbb{E}_{\theta}[\mathcal{J}^{\lambda^*}(\mathbf{a}(\cdot))], \qquad \forall\theta$$

Minimax Theorem:

$$\mathbb{E}_{\theta*}[\mathcal{J}^{\lambda^*}(\mathbf{a}(\cdot))] = \min_{\theta\in\Delta_{\mathcal{A}}} \max_{\lambda\in\Delta_r} \mathbb{E}_{\theta}[\mathcal{J}^{\lambda}(\mathbf{a}(\cdot))] = \max_{\lambda\in\Delta_r} \min_{\theta\in\Delta_{\mathcal{A}}} \mathbb{E}_{\theta}[\mathcal{J}^{\lambda}(\mathbf{a}(\cdot))]$$

λ - RESPONSE PROBLEM

Fix the observer's strategy $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, the optimal path of the evader $\boldsymbol{a}^{\boldsymbol{\lambda}}$ can be solved deterministically.

Path of $\boldsymbol{a}^{\lambda}$ when $\boldsymbol{\lambda} = (0.6, 0.4)$

Expected pointwise observability:

$$K^{\lambda}(\mathbf{x}, t) = \lambda_1 K_1(\mathbf{x}, t) + \lambda_2 K_2(\mathbf{x}, t)$$

Value function:

$$u^{\lambda}(\mathbf{x}) = \inf_{\mathbf{a}(\cdot)} \mathcal{J}^{\lambda}(\mathbf{x}, \mathbf{a}(\cdot))$$

Time-dependent HJB equation:

$$u_t^{\lambda}(\mathbf{x}) - \|\nabla u^{\lambda}(\mathbf{x})\| f(\mathbf{x}) + K^{\lambda}(\mathbf{x}, t) = 0$$

λ - RESPONSE PROBLEM

Fix the observer's strategy $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, the optimal path of the evader $\boldsymbol{a}^{\lambda}$ can be solved deterministically.

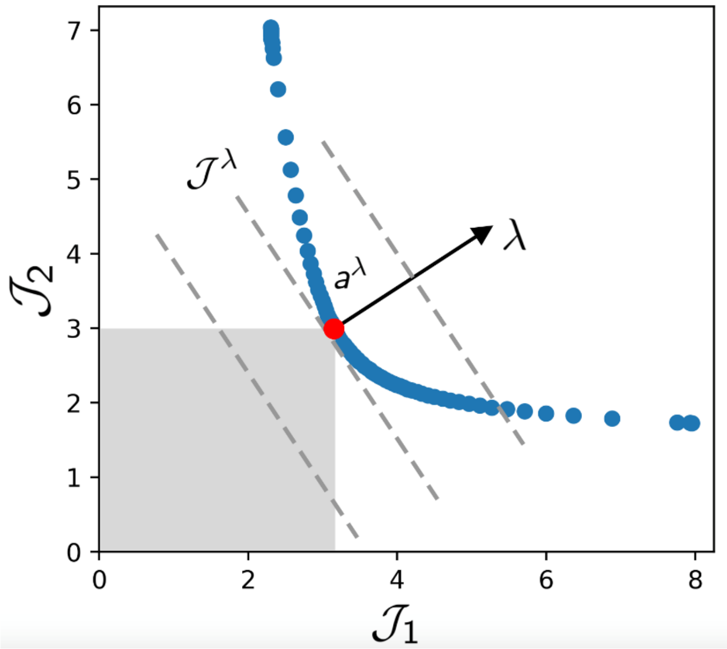
$\boldsymbol{\lambda} = (0.9,0.1)$

$\boldsymbol{\lambda} = (0.1,0.9)$

SCALARIZATION AND PARETO FRONT

Sample different λ's and plot $(\mathcal{J}_1(\boldsymbol{a}^{\lambda}), \mathcal{J}_2(\boldsymbol{a}^{\lambda}))$ to get the convex portion of the Pareto Front.

- Pareto-optimal strategy: not dominated by other strategies



$\boldsymbol{a}_1$ dominates $\boldsymbol{a}_2$:

$$\mathcal{J}_1(\boldsymbol{a}_1) \leq \mathcal{J}_1(\boldsymbol{a}_2)\ ,\ \mathcal{J}_2(\boldsymbol{a}_1) \leq \mathcal{J}_2(\boldsymbol{a}_2);$$

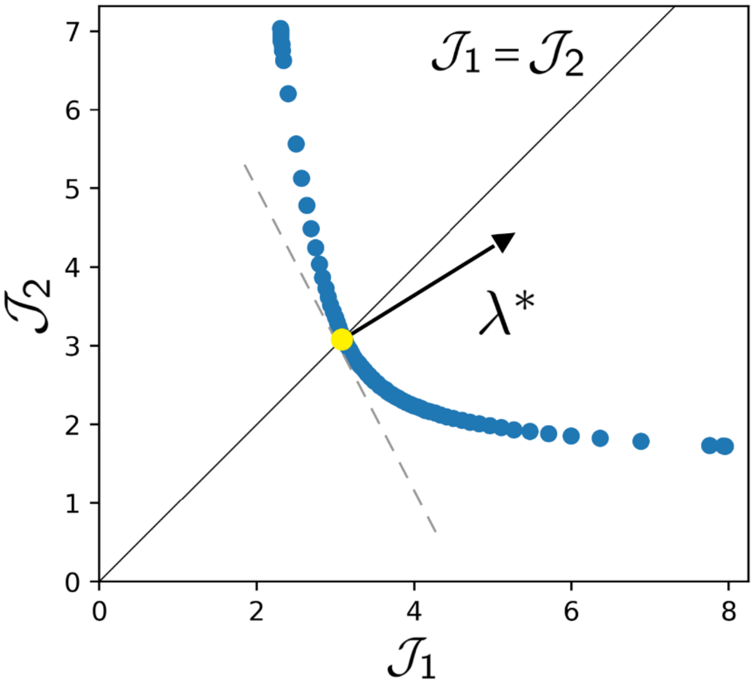
And at least one of the inequalities are strict.

$$\mathcal{J}^{\lambda}(\mathbf{a}(\cdot)) = \lambda_1 \mathcal{J}_1(\mathbf{a}(\cdot)) + \lambda_2 \mathcal{J}_2(\mathbf{a}(\cdot))$$

PURE STRATEGY NASH EQUILIBRIUM FOR EVADER

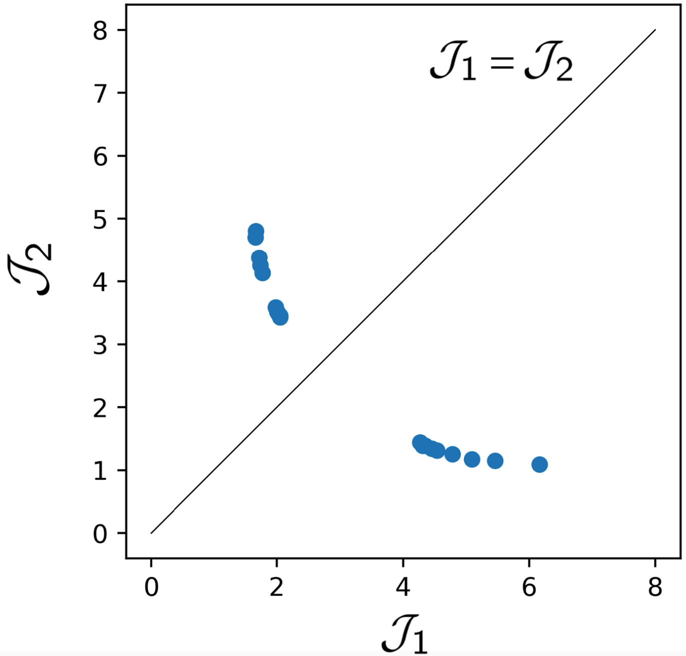
The $\boldsymbol{\lambda}^*$- optimal path $\boldsymbol{a}^{\boldsymbol{\lambda}^*}$ together with the probability distribution $\boldsymbol{\lambda}^*$ form a Nash equilibrium.

$$\mathcal{J}^{\lambda}(\mathbf{a}(\cdot)) = \lambda_1 \mathcal{J}_1(\mathbf{a}(\cdot)) + \lambda_2 \mathcal{J}_2(\mathbf{a}(\cdot))$$



MIXED STRATEGY NASH EQUILIBRIUM

A pure strategy Nash equilibrium does not always exist, but a mixed strategy Nash equilibrium always exists, and is attained at the minimax.



$$P(\lambda^*, \theta^*) = \min_{\theta \in \Delta_{\mathcal{A}}} \max_{\lambda \in \Delta_r} \mathbb{E}_\theta[\mathcal{J}^\lambda(\mathbf{a}(\cdot))] = \max_{\lambda \in \Delta_r} \min_{\theta \in \Delta_{\mathcal{A}}} \mathbb{E}_\theta[\mathcal{J}^\lambda(\mathbf{a}(\cdot))]$$

APPROXIMATION OF NASH EQUILIBRIUM – OBSERVER HALF

Find an approximate optimal strategy of the observer $\boldsymbol{\lambda}^*$ using convex optimization.

- (Gilles & Vladimirsky) Recall that $\mathcal{J}^{\lambda}(\mathbf{a}(\cdot)) = \lambda_1 \mathcal{J}_1(\mathbf{a}(\cdot)) + \lambda_2 \mathcal{J}_2(\mathbf{a}(\cdot))$
  Consider

$$\max_{\boldsymbol{\lambda} \in \Delta_r} \min_{\boldsymbol{a}(\cdot) \in \mathcal{A}} \mathcal{J}^{\boldsymbol{\lambda}}(\boldsymbol{x}_S, \boldsymbol{a}(\cdot)) = \max_{\boldsymbol{\lambda} \in \Delta_r} u^{\boldsymbol{\lambda}}(\boldsymbol{x}_S)$$

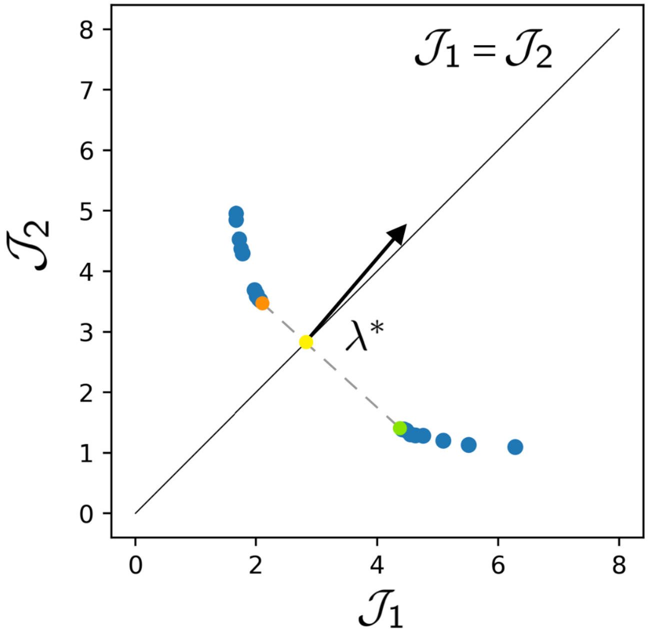
- Let $G(\boldsymbol{\lambda}) = \min_{\boldsymbol{a}(\cdot)\mathcal{A}} \sum_{i=1}^{r} \boldsymbol{\lambda}_i \mathcal{J}_i(\boldsymbol{a}(\cdot))$ and solve

$$\max_{\boldsymbol{\lambda}} G(\boldsymbol{\lambda})$$

$$\text{s.t.} \;\; \lambda_i \geq 0, \quad \sum_{i=1}^{r} \lambda_i = 1 \;.$$

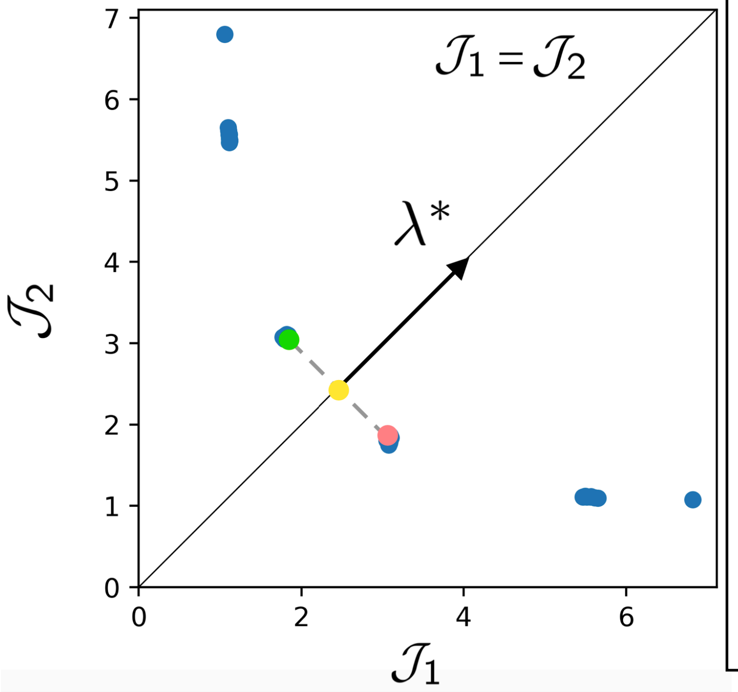
APPROXIMATION OF NASH EQUILIBRIUM – EVADER HALF

Find an approximate optimal strategy of the evader $\boldsymbol{\theta}^*$ by perturbing $\boldsymbol{\lambda}^*$ and adaptively growing the full set of pure strategies.



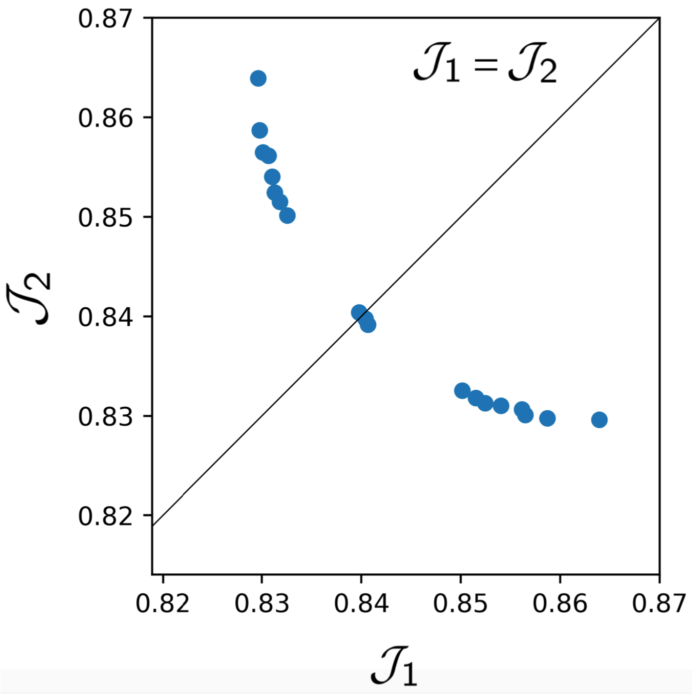
MIXED STRATEGY NASH EQUILIBRIUM

The convex portion of the Pareto Front does not intersect the central ray. In the mixed Nash equilibrium, the evader uses the two trajectories with probability $\boldsymbol{\theta}^* = (\theta_1, \theta_2)$.



ANISOTROPIC OBSERVERS

With anisotropic observers, the pointwise observability depends on the angle between the observer's direction of motion and the line of vision.



MULTIPLE EVADERS WITH CENTRAL PLANNER

Each evader E_l chooses a trajectory from his own source $\mathbf{x}_S^l$ to a single target $\mathbf{x}_T$ The goal now is to minimize the weighted sum of expected cumulative observabilities over all evaders.

Payoff function:

$$P(\boldsymbol{\lambda}, \{\boldsymbol{\theta}^l\}_{l=1}^q) = \sum_{l=1}^{l=q} w_l \mathbb{E}_\theta \left[\mathcal{J}^{l,\boldsymbol{\lambda}}(\boldsymbol{x}_S^l, \boldsymbol{a}^l(\cdot))\right]$$

Approximate $\boldsymbol{\lambda}^*$:

$$G^q(\boldsymbol{\lambda}) = \min_{\boldsymbol{a}^l(\cdot)} \sum_{l=1}^{l=q} w_l \mathcal{J}^{\boldsymbol{\lambda}}(\boldsymbol{x}_S^l, \boldsymbol{a}^l(\cdot))$$

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